REFORMULATION OF THE BROUWER GEOPOTENTIAL THEORY FOR IMPROVED COMPUTATIONAL EFFICIENCY

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Abstract. The theory, as derived by Brouwer and later modified by Lyddane, of the motion of an artificial Earth satellite perturbed by the first five zonal harmonics is reformulated in terms of an alternate set of variables. This alternate set of variables produces an equivalent solution, has no small eccentricity or small inclination restrictions, and allows calculation of position and velocity with considerably fewer algebraic and trigonometric operations. In addition, the alternate set of variables avoids one solution of Kepler's equation.

1. Introduction

A description of the motion of an artificial Earth satellite perturbed by the zonal harmonics J_2, J_3, J_4, and J_5 has been given by Brouwer (1959). He used the von Zeipel method to find a transformation of variables which significantly simplifies the transformed differential equations. Brouwer then found a solution of the transformed equations which, along with the transformation cited above, gives a solution containing all periodic terms through first order and all secular terms through second order, where J_2 is assumed to be first order and the remaining zonal harmonics are assumed to be second order. The theory is not applicable for small values of eccentricity or inclination or near the critical inclination.

Lyddane (1963) was able to reformulate the Brouwer transformation* in terms of an alternate set of variables such that the new transformation is also valid for small eccentricities and inclinations. Lyddane gives the transformation

$$x_1 = x_1'' + \delta x_1 \qquad x_2 = x_2'' + \delta x_2 \qquad x_3 = x_3'' + \delta x_3 \qquad (1)$$
$$x_4 = x_4'' + \delta x_4 \qquad x_5 = x_5'' + \delta x_5 \qquad x_6 = x_6'' + \delta x_6$$

where

$$x_1 = a \qquad x_2 = e \sin M \qquad x_3 = e \cos M \qquad x_4 = \sin I/2 \sin \Omega$$
$$x_5 = \sin I/2 \cos \Omega \qquad x_6 = M + \omega + \Omega \qquad (2)$$

with a = semimajor axis, e = eccentricity, I = inclination, M = mean anomaly, ω = argument of perigee, Ω = longitude of ascending node. The x_i'' have the same

* The Lyddane modification of the Brouwer theory will be referred to as the Brouwer–Lyddane theory.

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definitions as given by Equations (2) except that the quantities on the right-hand side should be replaced by the Brouwer double-primed elements* a'', e'', I'', M'', ω'', Ω''. Lyddane gives the first-order formulas

$$\begin{aligned}
\delta x_1 &= \delta a \\
\delta x_2 &= \delta e \sin M'' + e'' \, \delta M \cos M'' \\
\delta x_3 &= \delta e \cos M'' - e'' \, \delta M \sin M'' \\
\delta x_4 &= \tfrac{1}{2} \delta I \cos I''/2 \sin \Omega'' + \sin I''/2 \, \delta\Omega \cos \Omega'' \\
\delta x_5 &= \tfrac{1}{2} \delta I \cos I''/2 \cos \Omega'' - \sin I''/2 \, \delta\Omega \sin \Omega'' \\
\delta x_6 &= \delta M + \delta\omega + \delta\Omega
\end{aligned} \tag{3}$$

where δa, δe, δI, δM, $\delta\omega$, and $\delta\Omega$ are the terms of the Brouwer transformation which, as Lyddane points out, should be treated as the sum of the short- and long-period variations and strictly as functions of double-primed variables rather than the mixture which Brouwer uses.

The Brouwer–Lyddane theory forms the basis for many daily satellite tracking and prediction operations. To predict Cartesian position and velocity with the Brouwer–Lyddane theory, one must:

1. Solve Kepler's equation to find the double-primed true anomaly corresponding to the double-primed mean anomaly;
2. Apply the transformation given in Equations (1),
3. Inversely solve Equations (2) for the osculationg orbital elements;
4. Solve Kepler's equation to find the osculating true anomaly corresponding to the osculating mean anomaly; and
5. Compute position and velocity from the osculating orbital elements.

In many cases, operational considerations call for minimizing computer run-time. By defining an alternate set of variables, hereafter called position elements, we have found a convenient reformulation of the Brouwer–Lyddane transformation which avoids the second solution of Kepler's equation and the inverse solution of Equations (2) while still having no small eccentricity or small inclination restrictions. In addition, the position elements transformation requires only two trigonometric operations on the osculating variables in order to compute Cartesian position and velocity. Finally, the number of terms to be computed for the position elements transformation is smaller. In all, this formulation yields considerable computer run-time savings.

* The Brouwer formulas for predicting double-primed elements are given in the Appendix.

2. The Position Elements Transformation

The six position elements are defined by

$$y_1 = r = \frac{a\beta^2}{1 + e\cos f} \qquad y_2 = \dot{r} = \frac{nae}{\beta}\sin f \qquad y_3 = r\dot{f} = \frac{na^2\beta}{r}$$

$$y_4 = \sin I/2 \sin u \qquad y_5 = \sin I/2 \cos u \qquad y_6 = \lambda = f + \omega + \Omega \tag{4}$$

where

$$\beta = (1 - e^2)^{1/2} \qquad r = \text{radial distance}$$

$$n = \mu^{1/2} a^{-3/2} \qquad u = f + \omega$$

$$f = \text{true anomaly} \tag{5}$$

with μ being the product of Newton's gravitational constant and the mass of the Earth. Similarly, let the quantities y_1'', y_2'', y_3'', y_4'', y_5'', and y_6'' have the same definitions as given by Equations (4) and (5) except that the quantities on the right-hand sides should be replaced by the Brouwer double-primed elements.

By examining the functional dependence of the y_i on the x_j, we see that all order partial derivatives of the y_i with respect to the x_j exist, are continuous, and do not contain divisors of e or $\sin I$. Thus, we can expand the y_i in Taylor series about the point $(x_1'', x_2'', x_3'', x_4'', x_5'', x_6'')$ to obtain

$$y_i = y_i'' + \frac{\partial y_i}{\partial x_1}(x_1 - x_1'') + \frac{\partial y_i}{\partial x_2}(x_2 - x_2'') + \frac{\partial y_i}{\partial x_3}(x_3 - x_3'') +$$

$$+ \frac{\partial y_i}{\partial x_4}(x_4 - x_4'') + \frac{\partial y_i}{\partial x_5}(x_5 - x_5'') + \frac{\partial y_i}{\partial x_6}(x_6 - x_6'') + R_2, \tag{6}$$

where all partial derivatives are to be evaluated at the point $(x_1'', x_2'', x_3'', x_4'', x_5'', x_6'')$ and where R_2 denotes the remainder of the Taylor series. By calculating the second partial derivatives, we see that the next term in the series is of order $(J_2)^2$. It follows that R_2 can be dropped in Equation (6) to obtain a first-order transformation equation for the y_i which is valid for small eccentricities and small inclinations. The partial derivatives needed in Equation (6) can be calculated in a straightforward manner and are given by Hoots (1979). If we substitute Equations (3) and the appropriate partial derivatives into Equation (6), we find

$$y_1 - y_1'' = \frac{r}{a}\delta a - a\cos f\,\delta e + \frac{a}{\beta} e\,\delta M \sin f$$

$$y_2 - y_2'' = -\frac{1}{2}\frac{ne}{\beta}\sin f\,\delta a + na\beta\left(\frac{a}{r}\right)^2 \sin f\,\delta e + na\left(\frac{a}{r}\right)^2 e\,\delta M \cos f$$

$$y_3 - y_3'' = -\tfrac{1}{2} n\beta\left(\frac{a}{r}\right)\delta a + \frac{na}{\beta}\left(\frac{a}{r}\right)(\cos f - e + e\cos^2 f)\,\delta e -$$
$$- na\left(\frac{a}{r}\right)^2 e\,\delta M \sin f,$$
$$y_4 - y_4'' = \sin\frac{I}{2}\cos u\left[\frac{1}{\beta^2}(2 + e\cos f)\sin f\,\delta e + \right.$$
$$+ \frac{1}{\beta^3}(2 + e\cos f)\cos f\, e\delta M + \delta M + \delta\omega +$$
$$\left. + \frac{e}{\beta^3}\left(1 + \frac{\beta^2}{1+\beta}\right)e\delta M\right] + \frac{1}{2}\cos\frac{I}{2}\sin u\delta I, \tag{7}$$
$$y_5 - y_5'' = -\sin\frac{I}{2}\sin u\left[\frac{1}{\beta^2}(2 + e\cos f)\sin f\,\delta e + \right.$$
$$+ \frac{1}{\beta^3}(2 + e\cos f)\cos f\, e\delta M + \delta M + \delta\omega +$$
$$\left. + \frac{e}{\beta^3}\left(1 + \frac{\beta^2}{1+\beta}\right)e\delta M\right] + \frac{1}{2}\cos\frac{I}{2}\cos u\,\delta I,$$
$$y_6 - y_6'' = \frac{1}{\beta^2}(2 + e\cos f)\sin f\,\delta e + \delta M + \delta\omega + \delta\Omega +$$
$$+ \frac{1}{\beta^3}(2 + e\cos f)\cos f\, e\delta M + \frac{e}{\beta^3}\left(1 + \frac{\beta^2}{1+\beta}\right)e\delta M.$$

If we substitute the Brouwer transformation terms into the right-hand side of Equations (7), we have

$$\begin{aligned} y_1 - y_1'' &= \delta r_1 + \delta r_2 \\ y_2 - y_2'' &= \delta\dot r_1 + \delta\dot r_2 \\ y_3 - y_3'' &= \delta(r\dot f)_1 + \delta(r\dot f)_2 \\ y_4 - y_4'' &= \cos u''\sin I''/2(\delta u_1 + \delta u_2) + \tfrac{1}{2}\sin u''\cos I''/2(\delta I_1 + \delta I_2) \\ y_5 - y_5'' &= -\sin u''\sin I''/2\,(\delta u_1 + \delta u_2) + \tfrac{1}{2}\cos u''\cos I''/2(\delta I_1 + \delta I_2) \\ y_6 - y_6'' &= \delta\lambda_1 + \delta\lambda_2 \end{aligned} \tag{8}$$

where

$$\delta r_1 = -a\beta^2\sin I\left[C_1 e\sin I\cos(f + 2\omega) + \frac{1}{4}\frac{A_{3,0}}{k_2 a\beta^2}\sin(f+\omega) + \right.$$
$$\left. + C_4(4 + 3e^2)\sin(f+\omega) - C_5 e^2\sin(f + 3\omega) + 6C_4 e^2\sin f\cos\omega\right],$$

$$\delta r_2 = -\frac{1}{2}\frac{k_2}{a\beta^2}(-1+3\theta^2)\left[1+\frac{2r}{a\beta}+\frac{e\cos f}{1+\beta}\right]+ + \frac{1}{2}\frac{k_2}{a\beta^2}(1-\theta^2)\cos(2f+2\omega),$$

$$\delta \dot r_1 = na\beta^3\left(\frac{a}{r}\right)^2 \sin I\left[C_1 e\sin I\sin(f+2\omega) - \frac{1}{4}\frac{A_{3,0}}{k_2 a\beta^2}\cos(f+\omega) - - C_4(4+3e^2)\cos(f+\omega) + C_5 e^2\cos(f+3\omega) - 6C_4 e^2\cos f\cos\omega\right],$$

$$\delta \dot r_2 = \frac{1}{2}\frac{k_2 ne}{a\beta}(-1+3\theta^2)\left[\frac{1}{1+\beta}\left(\frac{a}{r}\right)^2+\frac{1}{\beta^3}\right]\sin f - - \frac{k_2 n}{a\beta}(1-\theta^2)\left(\frac{a}{r}\right)^2\sin(2f+2\omega),$$

$$\delta(r\dot f)_1 = -n\beta\left(\frac{a}{r}\right)^2\delta r_1 + na\beta\left(\frac{a}{r}\right)\frac{\sin I}{\theta}\,\delta I_1,$$

$$\delta(r\dot f)_2 = -n\beta\left(\frac{a}{r}\right)^2\delta r_2 + n\beta\left(\frac{a}{r}\right)\frac{\sin I}{\theta}\,\delta I_2,$$

$$\begin{aligned}\sin I''/2\delta u_1 = {} & \frac{1}{na\beta^3}\sin I/2\left(\frac{r}{a}\right)^2(2+e\cos f)\delta\dot r_1 + \\ & + \left[-\frac{1}{2}C_2 + C_3\theta^2\right]e^2\sin I/2\sin 2\omega - \frac{1}{8}\frac{A_{3,0}}{k_2 a\beta^2}\frac{e\theta^2}{\cos I/2}\cos\omega + \\ & + \frac{C_4 e}{2\cos I/2}(16-20\theta^2+6e^2-9e^2\theta^2)\cos\omega - \\ & - 6C_6 e\theta^2\sin I\sin I/2(4+3e^2)\cos\omega + \\ & + \tfrac{1}{6}C_5 e^3\frac{(-2+3\theta^2)}{\cos I/2}\cos 3\omega + \\ & + \tfrac{2}{3}C_7 e^3\theta^2\sin I\sin I/2\cos 3\omega,\end{aligned} \qquad (9)$$

$$\begin{aligned}\sin I''/2\delta u_2 = {} & \frac{1}{2}\frac{k_2}{a^2\beta^4}(-1+3\theta^2)(1-\beta)\left[\frac{e}{1+\beta}+\cos \mathrm{f}\right]\sin I/2\sin f + \\ & + \frac{1}{4}\frac{k_2}{a^2\beta^4}[(1-7\theta^2)\sin(2f+2\omega) + \\ & + 2e(2-5\theta^2)\sin(f+2\omega) - 2e\theta^2\sin(3f+2\omega)]\sin I/2 + \\ & + \frac{3}{2}\frac{k_2}{a^2\beta^4}(-1+5\theta^2)(f-M+e\sin f)\sin I/2,\end{aligned}$$

$$\delta I_1 = -e\theta\left[C_1 e \sin I \cos 2\omega + \frac{1}{4}\frac{A_{3,0}}{k_2 a\beta^2}\sin\omega + \right.$$
$$\left. + C_4(4+3e^2)\sin\omega - C_5 e^2 \sin 3\omega\right],$$
$$\delta I_2 = \frac{1}{2}\frac{k_2}{a^2\beta^4}\theta\sin I\left[3\cos(2f+2\omega) + 3e\cos(f+2\omega) + e\cos(3f+2\omega)\right],$$
$$\delta\lambda_1 = \frac{1}{na\beta^3}\left(\frac{r}{a}\right)^2(2+e\cos f)\delta\dot{r}_1 - \tfrac{1}{2}C_2 e^2\sin 2\omega -$$
$$- C_3 e^2\theta(1-\theta)\sin 2\omega + \frac{1}{4}\frac{A_{3,0}}{k_2 a\beta^2}\frac{e\theta}{1+\theta}\sin I\cos\omega + \tag{9}$$
$$+ \frac{e\sin I}{1+\theta}\left[C_4(16+20\theta+6e^2+9e^2\theta)\cos\omega + \right.$$
$$+ 6C_6\theta\sin^2 I(4+3e^2)\cos\omega -$$
$$\left. - \tfrac{1}{3}C_5 e^2(2+3\theta)\cos 3\omega - \tfrac{2}{3}C_7 e^2\theta\sin^2 I\cos 3\omega\right],$$
$$\delta\lambda_2 = \delta u_2 - \frac{1}{2}\frac{k_2}{a^2\beta^4}\theta[6(f-M+e\sin f) -$$
$$- 3\sin(2f+2\omega) - 3e\sin(f+2\omega) - e\sin(3f+2\omega)],$$

with

$$C_1 = \frac{1}{8}\frac{k_2}{a^2\beta^4}\frac{1}{(1-5\theta^2)}\left[(1-15\theta^2) - \frac{10}{3}\frac{k_4}{k_2^2}(1-7\theta^2)\right],$$
$$C_2 = C_1(1-\theta^2),$$
$$C_3 = \frac{1}{8}\frac{k_2}{a^2\beta^4}[11 + 80\theta^2(1-5\theta^2)^{-1} + 200\theta^4(1-5\theta^2)^{-2}] -$$
$$- \frac{5}{12}\frac{k_4}{k_2 a^2\beta^4}[3 + 16\theta^2(1-5\theta^2)^{-1} + 40\theta^4(1-5\theta^2)^{-2}],$$
$$C_4 = \frac{5}{64}\frac{A_{5,0}}{k_2 a^3\beta^6}[1 - 9\theta^2 - 24\theta^4(1-5\theta^2)^{-1}], \tag{10}$$
$$C_5 = \frac{35}{384}\frac{A_{5,0}}{k_2 a^3\beta^6}[1 - 5\theta^2 - 16\theta^4(1-5\theta^2)^{-1}],$$
$$C_6 = \frac{5}{64}\frac{A_{5,0}}{k_2 a^3\beta^6}[3 + 16\theta^2(1-5\theta^2)^{-1} + 40\theta^4(1-5\theta^2)^{-2}],$$
$$C_7 = \frac{35}{384}\frac{A_{5,0}}{k_2 a^3\beta^6}[5 + 32\theta^2(1-5\theta^2)^{-1} + 80\theta^4(1-5\theta^2)^{-2}],$$
$$\theta = \cos I,$$

where

$$k_2 = \frac{J_2}{2} R^2 \qquad A_{3,0} = -J_3 R^3$$

$$k_4 = -\tfrac{3}{8} J_4 R^4 \qquad A_{5,0} = -J_5 R^5 \qquad R = \text{equatorial radius.}$$

3. Cartesian Position and Velocity

Let **U** be a unit vector in the direction of the osculating radius vector and let **V** be a unit vector in the osculating orbital plane such that **U**, **V**, and the osculating angular momentum form a right-handed orthogonal system. Then

$$\begin{aligned}
U_x &= 2y_4(y_5 \sin y_6 - y_4 \cos y_6) + \cos y_6 \\
U_y &= -2y_4(y_5 \cos y_6 + y_4 \sin y_6) + \sin y_6 \\
U_z &= 2y_4 \cos I/2 \\
V_x &= 2y_5(y_5 \sin y_6 - y_4 \cos y_6) - \sin y_6 \\
V_y &= -2y_5(y_5 \cos y_6 + y_4 \sin y_6) + \cos y_6 \\
V_z &= 2y_5 \cos I/2
\end{aligned} \tag{11}$$

where $U_x, U_y, U_z, V_x, V_y, V_z$ denote the x, y, z components of the vectors **U** and **V**, respectively. The vectors **U** and **V** are expressed in terms of the osculating y_i except for the quantity $\cos I/2$ which is given by

$$\cos I/2 = \sqrt{1 - y_4^2 - y_5^2}. \tag{12}$$

Then the osculating position **r** and velocity $\dot{\mathbf{r}}$ are given by

$$\mathbf{r} = y_1 \mathbf{U} \qquad \dot{\mathbf{r}} = y_2 \mathbf{U} + y_3 \mathbf{V}. \tag{13}$$

4. Comparison With Numerical Integration

In order to obtain some numerical confirmation of the position elements transformation, a comparison was made with reference orbits generated with a Cowell numerical integration. Zonal harmonics through J_5 were included and the numerical integration was an eighth-order process with a variable integration step size.

The procedure for generating each test case was to choose an initial set of orbital elements and numerically integrate a reference orbit. This orbit was then assumed to represent the true integration of the equations of motion of the model described above. A best fit of the position elements solution to the reference orbit using a three-day data span was then found by a least-squares determination of the constants $a_0'', e_0'', I_0'', M_0'', \omega_0''$, and Ω_0''. The vector magnitude differences between the position elements theory and the numerical reference were then computed and listed in Table I.

TABLE I

Case number	a''_0 Earth radii	e''_0	I''_0 (deg.)	Position elements r.m.s. (m)	Brouwer–Lyddane r.m.s. (m)
1	1.2000	0.0000	0	23	26
2	1.2000	0.0000	45	7	10
3	1.2000	0.0000	90	10	12
4	1.2000	0.0100	0	23	26
5	1.2000	0.0100	45	10	14
6	1.2000	0.0100	90	10	14
7	1.2000	0.1000	0	24	29
8	1.2001	0.1000	45	9	12
9	1.2001	0.1001	90	10	14
10	2.1000	0.3000	0	21	24
11	2.1001	0.3000	45	19	21
12	2.1002	0.3000	90	17	19
13	2.1000	0.5000	0	23	36
14	2.0997	0.4999	45	10	14
15	2.0994	0.4999	90	7	17
16	4.0000	0.7000	0	21	36
17	3.9954	0.6997	45	9	10
18	3.9908	0.6993	90	7	19
19	10.1997	0.9000	0	64	120
20	9.9333	0.8973	45	19	21
21	9.6804	0.8946	90	21	57

For comparison, a best fit of the Brouwer–Lyddane theory to the reference orbit was also made and vector magnitude differences listed in Table I. Orbital characteristics are given in the table.

As can be seen from the table, not only does the position elements transformation agree very well with the reference orbits, but also in every case it performs slightly better than the Brouwer–Lyddane transformation. In addition, the position elements transformation saves 30–35% in computer run-time compared with the Brouwer–Lyddane transformation.

Appendix

$$a'' = a''_0 \qquad e'' = e''_0 \qquad I'' = I''_0$$

$$\begin{aligned} M'' = M''_0 + \Bigg\{ 1 &+ \frac{3}{2}\frac{k_2}{a_0^2\beta_0^3}(-1 + 3\theta_0^2) + \\ &+ \frac{3}{32}\frac{k_2^2}{a_0^4\beta_0^7}\Big[-15 + 16\beta_0 + 25\beta_0^2 + (30 - 96\beta_0 - 90\beta_0^2)\theta_0^2 + \\ &+ (105 + 144\beta_0 + 25\beta_0^2)\theta_0^4 \Big] + \\ &+ \frac{15}{16}\frac{k_4}{a_0^4\beta_0^7} e_0^2(3 - 30\theta_0^2 + 35\theta_0^4) \Bigg\} n_0(t - t_0), \end{aligned}$$

$$\omega = \omega_0'' + \left\{ \frac{3}{2}\frac{k_2}{a_0^2\beta_0^4}(-1+5\theta_0^2) + \frac{3}{32}\frac{k_2^2}{a_0^4\beta_0^8}[-35+24\beta_0+25\beta_0^2 + \right.$$

$$+(90-192\beta_0-126\beta_0^2)\theta_0^2+(385+360\beta_0+45\beta_0^2)\theta_0^4]+$$

$$+\frac{5}{16}\frac{k_4}{a_0^4\beta_0^8}[21-9\beta_0^2+(-270+126\beta_0^2)\theta_0^2+$$

$$\left. +(385-189\beta_0^2)\theta_0^4] \right\} n_0(t-t_0),$$

$$\Omega = \Omega_0'' + \left\{ -3\frac{k_2}{a_0^2\beta_0^4}\theta_0 + \frac{3}{8}\frac{k_2^2}{a_0^4\beta_0^8}[(-5+12\beta_0+9\beta_0^2)\theta_0 + \right.$$

$$\left. +(-35-36\beta_0-5\beta_0^2)\theta_0^3] + \frac{5}{4}\frac{k_4}{a_0^4\beta_0^8}(5-3\beta_0^2)\theta_0(3-7\theta_0^2) \right\} n_0(t-t_0),$$

where all quantities on the right-hand side are understood to be double-primed quantities and where the subscript 0 denotes the value at time t_0.

References

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