The variations δr, δi, $\delta \sin\theta$, $\delta\varphi$ can be readily incorporated into the computational scheme of section II.

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THE MOTION OF A CLOSE EARTH SATELLITE

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Abstract. In the present paper perturbations of six orbital elements of a close earth satellite moving in the gravitational field of the earth without air-resistance are derived as functions of mean orbital elements and time. No assumptions are made about the order of magnitude of eccentricity and inclination. However, it is assumed that the density distribution of the earth is symmetrical with respect to the axis of rotation, that the coefficient of the second harmonic of the potential is a small quantity of the first order and that those of the third and the fourth harmonics are of the second order. The results include periodic perturbations of the first order and secular perturbations up to the second order.

However, the solutions have some singularities for an orbit whose eccentricity or inclination is smaller than a quantity of the first order, and this case is treated in a different way.

By using Delaunay's canonical elements a theorem is proved that there are no long-periodic terms of the first order in the expression of the semi-major axis.

1. *The disturbing function.* In the present paper it is assumed that air-drag is absent and that the gravitational field of the earth is axially symmetric. Under these assumptions the gravitational potential of the earth at a point where the geocentric distance and the latitude are r and δ, respectively, is expanded into the series of spherical harmonics,

$$U = \frac{GM}{r}\left\{1 + \frac{A_2}{r^2}\left(\frac{1}{3} - \sin^2\delta\right) + \frac{A_3}{r^3}\left(\frac{5}{2}\sin^2\delta - \frac{3}{2}\right)\sin\delta + \frac{A_4}{r^4}\left(\frac{3}{35} + \frac{1}{7}\sin^2\delta - \frac{1}{4}\sin^2 2\delta\right) + \cdots\right\}, \quad (1)$$

where G is the gravitational constant and M is the mass of the earth. The second and fourth terms in the expression of the potential are due to the oblateness of the earth, and the third term is due to the asymmetry with respect to the equatorial plane. A_2 is taken to be of the first order of small quantities, and A_3 and A_4 are to be of the second order. The coefficients of higher harmonics may be of the third order of small quantities or less. (O'Keefe, Eckels and Squires 1959, Kozai 1959).

The purpose of the present author is to derive the periodic perturbations of the first order and secular perturbations up to the second order. Therefore terms of higher order than the fifth in the potential series may be neglected.

As the satellite is always on an ellipse, whose position and shape are variable, it is convenient to express r and δ in (1) by elliptical elements of the satellite by the following relations:

$$r = \frac{a(1 - e^2)}{1 + e\cos v}, \qquad \sin\delta = \sin i \sin(v + \omega), \quad (2)$$

where a is the semi-major axis, e is the eccentricity, i is the inclination to the equator, ω is the argument of perigee and v is the true anomaly.

The disturbing function due to the oblateness of the earth is then

$$R = U - GM/r$$

$$= GM\left[\frac{A_2}{a^3}\left(\frac{a}{r}\right)^3\left\{\frac{1}{3} - \frac{1}{2}\sin^2 i + \frac{1}{2}\sin^2 i\cos 2(v+\omega)\right\}\right.$$

$$+ \frac{A_3}{a^4}\left(\frac{a}{r}\right)^4\left\{\left(\frac{15}{8}\sin^2 i - \frac{3}{2}\right)\sin(v+\omega) - \frac{5}{8}\sin^2 i\sin 3(v+\omega)\right\}\sin i$$

$$+ \frac{A_4}{a^5}\left(\frac{a}{r}\right)^5\left\{\frac{3}{35} - \frac{3}{7}\sin^2 i + \frac{3}{8}\sin^4 i + \sin^2 i\left(\frac{3}{7} - \frac{1}{2}\sin^2 i\right)\cos 2(v+\omega)\right.$$

$$\left.\left. + \frac{1}{8}\sin^4 i\cos 4(v+\omega)\right\}\right]. \quad (3)$$

The true anomaly, v, is easily transformed to the mean anomaly, M, which is a linear function of time in non-perturbed motion, by the differential equation

$$\frac{dv}{dM} = \frac{a^2}{r^2}\sqrt{1 - e^2}; \quad (4)$$

r/a and v appearing in the disturbing function of (3) are then functions of e and M only, and are periodic with respect to M. Therefore, R is also a periodic function of M and ω. In R, terms depending neither on M nor on ω are called secular, terms depending on ω but not on M are long-periodic, and terms depending on M are short-periodic.

As the long-periodic perturbations originate from terms of the second order in R, we must retain secular terms and long-periodic terms up to the second order. However, for short-periodic terms we need only terms of the first order.

Using the following relations (Tisserand 1889):

$$\overline{\left(\frac{a}{r}\right)^3} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^3 dM = (1 - e^2)^{-3/2},$$

$$\overline{\left(\frac{a}{r}\right)^3\sin 2v} = \overline{\left(\frac{a}{r}\right)^3\cos 2v} = 0,$$

$$\overline{\left(\frac{a}{r}\right)^4\cos v} = e(1 - e^2)^{-5/2},$$

$$\overline{\left(\frac{a}{r}\right)^4\sin v} = \overline{\left(\frac{a}{r}\right)^4\cos 3v} = \overline{\left(\frac{a}{r}\right)^4\sin 3v} = 0, \quad (5)$$

$$\overline{\left(\frac{a}{r}\right)^5} = (1 - e^2)^{-7/2}\left(1 + \frac{3}{2}e^2\right),$$

$$\overline{\left(\frac{a}{r}\right)^5\cos 2v} = \frac{3}{4}e^2(1 - e^2)^{-7/2},$$

$$\overline{\left(\frac{a}{r}\right)^5\sin 2v} = \overline{\left(\frac{a}{r}\right)^5\cos 4v} = \overline{\left(\frac{a^5}{r}\right)\sin 4v} = 0,$$

we can pick up only necessary terms by the above criterion as follows:

$$R_1 = GM \frac{A_2}{a^3}\left(\frac{1}{3} - \frac{1}{2}\sin^2 i\right)(1 - e^2)^{-3/2},$$

$$R_2 = GM \frac{A_4}{a^5}\left(\frac{3}{35} - \frac{3}{7}\sin^2 i + \frac{3}{8}\sin^4 i\right)(1 - e^2)^{-7/2}\left(1 + \frac{3}{2}e^2\right),$$

$$R_3 = GM \left\{\frac{3}{2}\frac{A_3}{a^4}\sin i\left(\frac{5}{4}\sin^2 i - 1\right)e(1 - e^2)^{-5/2}\sin\omega + \frac{A_4}{a^5}\sin^2 i\left(\frac{9}{28} - \frac{3}{8}\sin^2 i\right)e^2(1 - e^2)^{-7/2}\cos 2\omega\right\}, \tag{6}$$

$$R_4 = GM \frac{A_2}{a^3}\left(\frac{a}{r}\right)^3\left[\left(\frac{1}{3} - \frac{1}{2}\sin^2 i\right)\left\{1 - \left(\frac{r}{a}\right)^3(1 - e^2)^{-3/2}\right\} + \frac{1}{2}\sin^2 i\cos 2(v + \omega)\right],$$

where R_1, R_2, R_3 and R_4 are first-order secular, second-order secular, long-periodic, and short-periodic parts of the disturbing function, respectively.

2. *Perturbations of short period.* The differential equations representing variations of orbital elements are:

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial M}, \\
\frac{de}{dt} &= \frac{1 - e^2}{na^2 e}\frac{\partial R}{\partial M} - \frac{\sqrt{1 - e^2}}{na^2 e}\frac{\partial R}{\partial \omega}, \\
\frac{d\omega}{dt} &= -\frac{\cos i}{na^2\sqrt{1 - e^2}\sin i}\frac{\partial R}{\partial i} + \frac{\sqrt{1 - e^2}}{na^2 e}\frac{\partial R}{\partial e}, \\
\frac{di}{dt} &= \frac{\cos i}{na^2\sqrt{1 - e^2}\sin i}\frac{\partial R}{\partial \omega}, \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1 - e^2}\sin i}\frac{\partial R}{\partial i}, \\
\frac{dM}{dt} &= n - \frac{1 - e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a},
\end{aligned} \tag{7}$$

where n is related to a by $n^2a^3 = GM$.

To derive short-periodic perturbations of the first order, after replacing R by R_4 in (7), one may regard a, n, e, i and ω on the right-hand sides of equations (7) to be constant. However, n, appearing as the first term of the last equation without any factor, is variable, but it is a known function of time, after obtaining the expression of the semi-major axis. True anomaly, v, may be regarded also to be a known function of time on the right-hand side, and one can transform the independent variable from t to v by

$$dt = \frac{dt}{dM}dM = \frac{1}{n}\left(\frac{r}{a}\right)^2\frac{1}{\sqrt{1 - e^2}}dv. \tag{8}$$

Then, for example, the short-periodic perturbations of the inclination are obtained by,

$$di_s = \frac{\cos i}{n^2a^2(1 - e^2)\sin i}\int\left(\frac{r}{a}\right)^2\frac{\partial R_4}{\partial \omega}dv, \tag{9}$$

where the integrand can be expressed by a finite trigonometric series and is, of course, integrable analytically within the necessary accuracy. The results for the six elements are as follows:

$$da_s = \frac{A_2}{a}\left[\frac{2}{3}\left(1-\frac{3}{2}\sin^2 i\right)\left\{\left(\frac{a}{r}\right)^3-(1-e^2)^{-3/2}\right\}+\left(\frac{a}{r}\right)^3\sin^2 i\cos 2(v+\omega)\right],$$

$$de_s = \frac{1-e^2}{e}\frac{A_2}{a^2}\left[\frac{1}{3}\left(1-\frac{3}{2}\sin^2 i\right)\left\{\left(\frac{a}{r}\right)^3-(1-e^2)^{-3/2}\right\}+\frac{1}{2}\left(\frac{a}{r}\right)^3\sin^2 i\cos 2(v+\omega)\right]$$
$$-\frac{\sin^2 i}{2e}\frac{A_2}{ap}\left\{\cos 2(v+\omega)+e\cos(v+2\omega)+\frac{1}{3}e\cos(3v+2\omega)\right\},$$

$$d\omega_s = \frac{A_2}{p^2}\left[\left(2-\frac{5}{2}\sin^2 i\right)(v-M+e\sin v)\right.$$
$$+\left(1-\frac{3}{2}\sin^2 i\right)\left\{\frac{1}{e}\left(1-\frac{1}{4}e^2\right)\sin v+\frac{1}{2}\sin 2v+\frac{e}{12}\sin 3v\right\}$$
$$-\frac{1}{e}\left\{\frac{1}{4}\sin^2 i+\left(\frac{1}{2}-\frac{15}{16}\sin^2 i\right)e^2\right\}\sin(v+2\omega)+\frac{e}{16}\sin^2 i\sin(v-2\omega)$$
$$-\frac{1}{2}\left(1-\frac{5}{2}\sin^2 i\right)\sin 2(v+\omega)+\frac{1}{e}\left\{\frac{7}{12}\sin^2 i-\frac{1}{6}\left(1-\frac{19}{8}\sin^2 i\right)e^2\right\}\sin(3v+2\omega) \quad (10)$$
$$\left.+\frac{3}{8}\sin^2 i\sin(4v+2\omega)+\frac{e}{16}\sin^2 i\sin(5v+2\omega)\right],$$

$$di_s = \frac{1}{4}\frac{A_2}{p^2}\sin 2i\left\{\cos 2(v+\omega)+e\cos(v+2\omega)+\frac{e}{3}\cos(3v+2\omega)\right\},$$

$$d\Omega_s = -\frac{A_2}{p^2}\cos i\left\{v-M+e\sin v-\frac{1}{2}\sin 2(v+\omega)-\frac{e}{2}\sin(v+2\omega)-\frac{e}{6}\sin(3v+2\omega)\right\},$$

$$e\,dM_s = \frac{A_2}{p^2}\sqrt{1-e^2}\left[-\left(1-\frac{3}{2}\sin^2 i\right)\left\{\left(1-\frac{e^2}{4}\right)\sin v+\frac{e}{2}\sin 2v+\frac{e^2}{12}\sin 3v\right\}\right.$$
$$+\sin^2 i\left\{\frac{1}{4}\left(1+\frac{5}{4}e^2\right)\sin(v+2\omega)-\frac{e^2}{16}\sin(v-2\omega)-\frac{7}{12}\left(1-\frac{e^2}{28}\right)\sin(3v+2\omega)\right.$$
$$\left.\left.-\frac{3}{8}e\sin(4v+2\omega)-\frac{e^2}{16}\sin(5v+2\omega)\right\}\right],$$

where

$$p = a(1-e^2).$$

As the mean values of $\cos jv$ $(j = 1, 2, \cdots)$ with respect to M do not vanish, but

$$\overline{\cos jv} = \left(\frac{-e}{1+\sqrt{1-e^2}}\right)^j (1+j\sqrt{1-e^2}), \quad (11)$$

mean values of these short-periodic perturbations are not zero, except for those of a. Their mean values with respect to M are:

$$
\begin{aligned}
\overline{de_s} &= \frac{A_2}{p^2}\sin^2 i\,\frac{1-e^2}{6e}\,\overline{\cos 2v}\cos 2\omega,\\
\overline{d\omega_s} &= \frac{A_2}{p^2}\left\{\sin^2 i\left(\frac{1}{8}+\frac{1-e^2}{6e^2}\,\overline{\cos 2v}\right)+\frac{1}{6}\cos^2 i\,\overline{\cos 2v}\right\}\sin 2\omega,\\
\overline{di_s} &= -\frac{1}{12}\frac{A_2}{p^2}\sin 2i\,\overline{\cos 2v}\cos 2\omega,\\
\overline{d\Omega_s} &= -\frac{1}{6}\frac{A_2}{p^2}\cos i\,\overline{\cos 2v}\sin 2\omega,\\
\overline{dM_s} &= -\frac{A_2}{p^2}\sqrt{1-e^2}\sin^2 i\left\{\frac{1}{8}+\frac{1+\frac{e^2}{2}}{6e^2}\,\overline{\cos 2v}\right\}\sin 2\omega.
\end{aligned}
\tag{12}
$$

Therefore, $de_s - \overline{de_s}$, $d\omega_s - \overline{d\omega_s}$, $di_s - \overline{di_s}$, $dM_s - \overline{dM_s}$ and $d\Omega_s - \overline{d\Omega_s}$ must be the short-periodic perturbations whose mean values with respect to the mean anomaly are zero.

Expressions of the mean anomaly and the argument of perigee are rather complicated, so it is better to combine the four elements a, e, ω and M into the radius vector r and the argument of latitude $L = v + \omega$ by the following relations:

$$
\begin{aligned}
\frac{dr}{a} &= \frac{e}{\sqrt{1-e^2}}\sin v\,dM + \frac{r}{a}\frac{da}{a} - \cos v\,de,\\
dv &= \frac{a^2}{r^2}\sqrt{1-e^2}\,dM + \sin v\left(1+\frac{r}{p}\right)\frac{a}{r}\,de.
\end{aligned}
$$

Putting

$$
\begin{aligned}
da &= -a_0\frac{A_2}{p^2}\left(1-\frac{3}{2}\sin^2 i\right)\sqrt{1-e^2} + da_s, \qquad de = de_s,\\
d\omega &= d\omega_s - \frac{3}{8}\frac{A_2}{p^2}\sin^2 i\sin 2\omega,\\
dM &= dM_s + \frac{3}{8}\frac{A_2}{p^2}\sqrt{1-e^2}\sin^2 i\sin 2\omega,
\end{aligned}
$$

the deviations of the radius vector and argument of latitude of the satellite, from those computed by mean orbital elements are obtained as follows:

$$
\begin{aligned}
\frac{dr}{a} &= \frac{1}{3}\frac{A_2}{ap}\left(1-\frac{3}{2}\sin^2 i\right)\left\{-1-\frac{1}{e}\left(1-\sqrt{1-e^2}\right)\cos v + \frac{r}{a}\frac{1}{\sqrt{1-e^2}}\right\} + \frac{1}{6}\frac{A_2}{ap}\sin^2 i\cos 2(v+\omega),\\
dL &= \frac{A_2}{p^2}\Bigg[\left(2-\frac{5}{2}\sin^2 i\right)(v - M + e\sin v)\\
&\quad + \left(1-\frac{3}{2}\sin^2 i\right)\left\{\frac{2}{3e}\left(1-\frac{e^2}{2}-\sqrt{1-e^2}\right)\sin v + \frac{1}{6}\left(1-\sqrt{1-e^2}\right)\sin 2v\right\}\\
&\quad - \left(\frac{1}{2}-\frac{5}{6}\sin^2 i\right)e\sin(v+2\omega) - \left(\frac{1}{2}-\frac{7}{12}\sin^2 i\right)\sin 2(v+\omega) - \frac{e}{6}\cos^2 i\sin(3v+2\omega)\Bigg].
\end{aligned}
\tag{13}
$$

The secular perturbations of the first order are easily derived by putting $R = R_1$ in (7), as

$$\bar{\omega} = \omega_0 + \frac{A_2}{p^2}\bar{n}\left(2 - \frac{5}{2}\sin^2 i\right)t,$$

$$\bar{\Omega} = \Omega_0 - \frac{A_2}{p^2}\bar{n}t\cos i$$

$$\bar{M} = M_0 + \bar{n}t,$$

$$\bar{n} = n_0 + \frac{A_2}{p^2}n_0\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1 - e^2},$$

where ω_0, Ω_0 and M_0 are mean values at the epoch, that is, the initial values, from which periodic perturbations have been subtracted. n_0 is the unperturbed mean motion, which is related to the unperturbed semi-major axis a_0 by $n_0^2 a_0^3 = GM$.

It is more convenient to adopt as a mean value of the semi-major axis not a_0 but

$$\bar{a} = a_0\left\{1 - \frac{A_2}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1 - e^2}\right\},$$

so that the following relation holds:

$$\bar{n}^2\bar{a}^3 = GM\left\{1 - \frac{A_2}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1 - e^2}\right\}. \tag{14}$$

And to derive the expressions of (13) this value of $\bar{a}$ has been already adopted as a mean semi-major axis.

3. *A theorem on non-existence of long-periodic terms in the semi-major axis.* As is well known, Poisson proved a theorem of non-existence of secular perturbations of the semi-major axis of a planetary orbit. However, there are some differences between theories of planetary and satellite motions. Because the perihelion and node of an ordinary planet moves around the sun with the period of some ten thousand years, one may expand $e\sin\omega$, $e\cos\omega$, $i\sin\Omega$ and $i\cos\Omega$ into power series of time in the planetary theory, so that terms depending only on ω and Ω must be regarded to be secular, not long-periodic. From this point of view Poisson proved the theorem. Now we observe the motion of an earth satellite for a long interval of time, in which the line of apside makes several revolutions. So the corresponding theorem in the satellite motion is that there are no long-periodic terms in the expression of the semi-major axis.

Let us transform variables from Kepler's elements to Delaunay's canonical ones:

$$\begin{cases} L = \sqrt{\mu a}, & G = \sqrt{\mu a(1 - e^2)}, & H = \sqrt{\mu a(1 - e^2)}\cos i, \\ l = M, & g = \omega, & h = \Omega, \end{cases} \tag{15}$$

where $\mu = GM$.

These variables must satisfy the following canonical equations:

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}, & \frac{dG}{dt} &= \frac{\partial F}{\partial g}, & \frac{dH}{dt} &= \frac{\partial F}{\partial h}, \\ \frac{dl}{dt} &= -\frac{\partial F}{\partial L}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G}, & \frac{dh}{dt} &= -\frac{\partial F}{\partial H}, \end{aligned} \tag{16}$$

where

$$F = \frac{\mu^2}{2L^2} + R.$$

Since F does not depend on h, there is an integral H = constant. Variations of the semi-major axis are represented by:

$$\frac{dL}{dt} = \frac{\partial F}{\partial l} = \frac{\partial R_4}{\partial l},$$

where $\mu^2/2L^2$, R_1, R_2 and R_3 are omitted in F because they do not depend on l.

Long-periodic perturbations of the first order of L will come from long-periodic terms of the second order of $\partial R_4/\partial l$, which may be expanded into Taylor's series,

$$\frac{\partial R_4}{\partial l} = \left(\frac{\partial R_4}{\partial l}\right) + \left(\frac{\partial^2 R_4}{\partial l \partial L}\right) dL + \left(\frac{\partial^2 R_4}{\partial l^2}\right) dl + \left(\frac{\partial^2 R_4}{\partial l \partial G}\right) dG + \left(\frac{\partial^2 R_4}{\partial l \partial g}\right) dg + \cdots \tag{17}$$

where terms of higher than the third order are neglected and in parentheses l, L, g and G are replaced by their respective mean values, $l = \bar{n}t + l_0$, $L = \bar{L}_0$, $g = \dot{g}t + g_0$ and $G = \bar{G}$; dl, dL, dG and dg are deviations of instantaneous values from their respective mean values and are of the first order. $\bar{n}$ and $\dot{g}$ are secular motions of the mean longitude and the longitude of perigee, and are supposed to be known.

In equation (17) one must consider only long-periodic terms of the second order, and after integrating with respect to time only such terms of the first order, omiting other terms. The first term of (17), $\partial R_4/\partial l$, cannot have any long-periodic terms of any order, so must be omitted. As R_4 depends on t only through M within the accuracy of the first order, the following relations hold:

$$\begin{aligned} &\int\left(\frac{\partial^2 R_4}{\partial l^2}\right) dt = \frac{1}{\bar{n}}\left(\frac{\partial R_4}{\partial l}\right), \qquad \int\left(\frac{\partial^2 R_4}{\partial l \partial L}\right) dt = \frac{1}{\bar{n}}\left(\frac{\partial R_4}{\partial L}\right), \\ &\int\left(\frac{\partial^2 R_4}{\partial l \partial G}\right) dt = \frac{1}{\bar{n}}\left(\frac{\partial R_4}{\partial G}\right), \qquad \int\left(\frac{\partial^2 R_4}{\partial l \partial g}\right) dt = \frac{1}{\bar{n}}\left(\frac{\partial R_4}{\partial g}\right). \end{aligned} \tag{18}$$

Integrating $\partial R_4/\partial l$ in (17) with respect to time by parts using the relations (18), one has

$$\begin{aligned} dL &= \int \frac{\partial R_4}{\partial l} dt \\ &= \frac{1}{\bar{n}}\left[\left(\frac{\partial R_4}{\partial L}\right) dL + \left(\frac{\partial R_4}{\partial l}\right) dl + \left(\frac{\partial R_4}{\partial G}\right) dG + \left(\frac{\partial R_4}{\partial g}\right) dg\right] \\ &\qquad - \frac{1}{\bar{n}}\int\left\{\left(\frac{\partial R_4}{\partial L}\right)\frac{dL}{dt} + \left(\frac{\partial R_4}{\partial l}\right)\left(\frac{dl}{dt} - \bar{n}\right) + \left(\frac{\partial R_4}{\partial G}\right)\frac{dG}{dt} + \left(\frac{\partial R_4}{\partial g}\right)\left(\frac{dg}{dt} - \dot{g}\right)\right\} dt. \end{aligned} \tag{19}$$

After integration the first part contains only terms of the second order, so can be omitted. The second part is transformed to

$$\begin{aligned} dL &= -\frac{1}{\bar{n}}\int\left[\left(\frac{\partial R_4}{\partial L}\right)\left(\frac{\partial R_4}{\partial l}\right) + \left(\frac{\partial R_4}{\partial l}\right)\left\{dn - \left(\frac{\partial R_4}{\partial L}\right)\right\}\right. \\ &\qquad\qquad \left. + \left(\frac{\partial R_4}{\partial G}\right)\left(\frac{\partial R_4}{\partial g}\right) + \left(\frac{\partial R_4}{\partial g}\right)\left(\frac{\partial R_4}{\partial G}\right)\right] dt \\ &= -\frac{1}{\bar{n}}\int dn \left(\frac{\partial R_4}{\partial l}\right) dt. \end{aligned} \tag{20}$$

By using the relation nL^3 = const., this equation is written as

$$dL = \frac{3}{L}\int\left(\frac{\partial R_4}{\partial l}\right) dLdt$$

$$= \frac{3}{\bar{n}L}[R_4dL] - \frac{3}{\bar{n}L}\int (R_4)\frac{dL}{dt}dt$$

$$= \frac{3}{\bar{n}L}[R_4dL] - \frac{3}{\bar{n}L}\int (R_4)\left(\frac{\partial R_4}{\partial l}\right)dt$$

$$= \frac{3}{\bar{n}L}[R_4dL] - \frac{3}{2\bar{n}L}\int\left(\frac{\partial R_4^2}{\partial l}\right)dt$$

$$= \frac{3}{\bar{n}L}[R_4dL] - \frac{3}{2\bar{n}^2L}(R_4^2). \tag{21}$$

Since the terms are all of the second order after integration, one can conclude that there are no long-periodic perturbations of the first order in the expression of the semi-major axis. Of course there are no secular terms in the semi-major axis.

4. *Secular perturbations of the second order and long-periodic perturbations.* Let E_i be one of the six orbital elements of a satellite, and express its variation by the differential equation

$$\frac{dE_i}{dt} = f_i. \tag{22}$$

A function f_i may be expanded into a power series of deviations from the mean orbital elements as

$$\frac{dE_i}{dt} = (f_i) + \sum_j \left(\frac{\partial f_i}{\partial E_j}\right) dE_j + \cdots. \tag{23}$$

Secular perturbations of the second order and long-periodic perturbations of the first order will come from the terms of the second order on the right-hand side of (23). However, the complete first order expression of dE_j is not yet known except for da. By integrating equation (23) by parts as in the previous section, there holds,

$$dE_i = \int (f_i)dt + \sum_j [F_{ij}dE_j] - \sum_j \int F_{ij}\frac{dE_j}{dt}dt, \tag{24}$$

where

$$F_{ij} = \int\left(\frac{\partial f_i}{\partial E_j}\right)dt.$$

If E_i is the inclination, i, for example,

$$f_i = \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \omega}.$$

In this case $\partial f_i/\partial E_j$ has neither long-periodic nor secular terms as far as the terms of the first order are concerned. As it is easily proved that i and e cannot contain any secular terms, $\sum_j F_{ij}dE_j$ can be omitted from the equation of the inclination, because this is of the second order and has no secular terms.

Therefore all terms on the right-hand side of (23) for the inclination are known and long-periodic terms can be picked up by using the following relations (Tisserand 1889):

$$\overline{\left(\frac{a}{r}\right)^6} = (1-e^2)^{-9/2}\left(1+3e^2+\frac{3}{8}e^4\right),$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos v} = 2e(1-e^2)^{-9/2}\left(1+\frac{3}{4}e^2\right),$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos 2v} = \frac{3}{2}e^2(1-e^2)^{-9/2}\left(1+\frac{1}{6}e^2\right),$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos 3v} = \frac{e^3}{2}(1-e^2)^{-9/2},$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos 4v} = \frac{e^4}{16}(1-e^2)^{-9/2},$$

$$\overline{\left(\frac{a}{r}\right)^6 \cos jv} = 0, \quad (j>4),$$

$$\overline{\left(\frac{a}{r}\right)^3} = (1-e^2)^{-3/2},$$

$$\overline{\left(\frac{a}{r}\right)^3 \cos v} = \frac{e}{2}(1-e^2)^{-3/2},$$

$$\overline{\left(\frac{a}{r}\right)^3 \cos jv} = 0, \quad (j>1).$$

Then the expression of long-periodic perturbations of the inclination are derived as:

$$di_1 = \overline{di_s} - \frac{A_2}{p^2}\frac{e^2 \sin 2i}{8(4-5\sin^2 i)}\left\{\frac{14-15\sin^2 i}{6} - \frac{A_4}{A_2^2}\frac{18-21\sin^2 i}{7}\right\}\cos 2\omega - \frac{3}{4}\frac{A_3}{A_2 p}e\cos i \sin\omega. \quad (25)$$

Now there is an integral, $\sqrt{a(1-e^2)}\cos i = \text{const.}$, and it has been already proved that there are no long-periodic perturbations of the first order in the semi-major axis, so de_1, the long-periodic perturbation of the eccentricity, is

$$de_1 = -\frac{1-e^2}{e}\frac{\sin i}{\cos i}di_1$$

$$= \overline{de_s} + \frac{A_2}{pa}\frac{e\sin^2 i}{4(4-5\sin^2 i)}\left\{\frac{14-15\sin^2 i}{6} - \frac{A_4}{A_2^2}\frac{18-21\sin^2 i}{7}\right\}\cos 2\omega + \frac{3}{4}\frac{A_3}{A_2 a}\sin i \sin\omega. \quad (26)$$

However, the same principle cannot be applied to obtain the expressions of the node and the argument of perigee, because in these cases $\partial f_i/\partial i$, $\partial f_i/\partial e$ and $\partial f_i/\partial a$ have secular terms. But fortunately complete expressions of di, de and da have already been derived. And since it is also proved that $F_{j\omega}d\omega$ and $F_{jM}dM$ do not include secular terms, the following expressions are derived:

$$\dot{\Omega} = -\frac{A_2}{\bar{p}^2}\bar{n}\cos\bar{i}\left[1+\frac{A_2}{p^2}\left\{\frac{3}{2}+\frac{e^2}{6}-2\sqrt{1-e^2}-\sin^2 i\left(\frac{5}{3}-\frac{5}{24}e^2-3\sqrt{1-e^2}\right)\right\}\right]$$

$$-\frac{A_4}{p^4}n\cos i\,\frac{12-21\sin^2 i}{14}\left(1+\frac{3}{2}e^2\right), \quad (27)$$

$$\dot{\omega} = \frac{A_2}{\bar{p}^2}\bar{n}\left(2 - \frac{5}{2}\sin^2\bar{\imath}\right)\left[1 + \frac{A_2}{p^2}\left\{2 + \frac{e^2}{2} - 2\sqrt{1-e^2} - \sin^2 i\left(\frac{43}{24} - \frac{e^2}{48} - 3\sqrt{1-e^2}\right)\right\}\right] - \frac{5}{12}\frac{A_2^2}{p^4}e^2 n\cos^4 i$$
$$+ \frac{A_4}{p^4}n\left[\frac{12}{7} - \frac{93}{14}\sin^2 i + \frac{21}{4}\sin^4 i + e^2\left(\frac{27}{14} - \frac{189}{28}\sin^2 i + \frac{81}{16}\sin^4 i\right)\right], \quad (28)$$

where $\bar{\imath}$ and $\bar{e}$ are mean values of the inclination and the eccentricity over all the periods, and

$$\bar{p} = \bar{a}(1 - \bar{e}^2).$$

$$d\Omega_1 = \overline{d\Omega_s} - \frac{A_2}{p^2}\frac{e^2\cos i}{2(4-5\sin^2 i)}\left[\left\{\frac{7-15\sin^2 i}{6} - \frac{A_4}{A_2^2}\frac{9-21\sin^2 i}{7}\right\}\right.$$
$$\left. + \frac{5\sin^2 i}{2(4-5\sin^2 i)}\left\{\frac{14-15\sin^2 i}{6} - \frac{A_4}{A_2^2}\frac{18-21\sin^2 i}{7}\right\}\right]\sin 2\omega + \frac{3}{4}\frac{A_3}{A_2 p}\frac{\cos i}{\sin i}e\cos\omega, \quad (29)$$

$$d\omega_1 = \overline{d\omega_s} - \frac{3}{8}\frac{A_2}{p^2}\sin^2 i\sin 2\omega$$
$$- \frac{A_2}{p^2}\left[\frac{1}{4-5\sin^2 i}\left\{\frac{14-15\sin^2 i}{24}\sin^2 i - e^2\frac{28-158\sin^2 i + 135\sin^4 i}{48}\right.\right.$$
$$\left. - \frac{A_4}{A_2^2}\left(\frac{18-21\sin^2 i}{28}\sin^2 i - e^2\frac{36-210\sin^2 i + 189\sin^4 i}{56}\right)\right\}$$
$$\left. - \frac{e^2\sin^2 i(13-15\sin^2 i)}{(4-5\sin^2 i)^2}\left(\frac{14-15\sin^2 i}{24} - \frac{A_4}{A_2^2}\frac{18-21\sin^2 i}{28}\right)\right]\sin 2\omega$$
$$+ \frac{3}{4}\frac{A_3}{A_2 p}\frac{\sin^2 i - e^2\cos^2 i}{\sin i}\frac{1}{e}\cos\omega. \quad (30)$$

It is very difficult to derive the long-periodic perturbations of the first order of the mean anomaly because the long-periodic perturbations of the second order of the semi-major axis are not known. However, usually the atmosphere of the earth changes the mean motion of the actual satellite so much that any long-periodic variations of the mean anomaly cannot be detected from observations with good accuracy.

The results are then:

$$a = \bar{a} + da_s, \qquad \bar{a} = a_0\left\{1 - \frac{A_2}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right\},$$
$$e = \bar{e} + de_s - \overline{de_s} + de_1,$$
$$i = \bar{\imath} + di_s - \overline{di_s} + di_1,$$
$$\omega = \omega_0 + \dot{\omega}t + d\omega_s - \overline{d\omega_s} + d\omega_1, \quad (31)$$
$$\Omega = \Omega_0 + \dot{\Omega}t + d\Omega_s - \overline{d\Omega_s} + d\Omega_1,$$
$$M = M_0 + \bar{n}t + dM_s, \qquad \bar{n} = n_0\left\{1 + \frac{A_2}{p^2}\left(1 - \frac{3}{2}\sin^2 i\right)\sqrt{1-e^2}\right\}.$$
$$n_0^2 a_0^3 = GM,$$

where $\bar{e}$ and $\bar{\imath}$ are mean values with respect to M and ω, and ω_0, Ω_0 and M_0 are initial values from which periodic perturbations have been subtracted. On the right-hand sides of these expressions one must always use constant mean values as a, e, i, $\omega - \dot{\omega}t$, $\Omega - \dot{\Omega}t$ and $M - \bar{n}t$.

5. *The case when the inclination or eccentricity is very small.* In the previous sections it is assumed that $4-5\sin^2 i$, i and e are not very small. If e is very small, periodic perturbations of M and ω become very large. However, in the case of the short-periodic perturbations the difficulty will vanish, if elements are transformed from M and ω to r and $v + \omega$.

For the long-periodic perturbations the variables must be transformed from e and ω to

$$\xi = e\cos\omega,$$

$$\eta = -e\sin\omega.$$

These variables must satisfy

$$\frac{d\xi}{dt} = \frac{\partial R}{\partial\eta} = \dot{\omega}\eta - \frac{3}{2}\frac{A_3}{a^3}\sin i\left(\frac{5}{4}\sin^2 i - 1\right),$$
$$\frac{d\eta}{dt} = -\frac{\partial R}{\partial\xi} = -\dot{\omega}\xi, \qquad (32)$$

where $\dot{\omega}$ has the same value as in the previous section.

The solutions of these equations are

$$e\cos\omega = e_0\cos\bar{\omega},$$

$$e\sin\omega = e_0\sin\bar{\omega} + \frac{3}{4}\frac{A_3}{aA_2}\sin i,$$

where

$$\bar{\omega} = \dot{\omega}t + \omega_0.$$

e_0 and ω_0 are constants of integration.

If e_0 is much smaller than $3A_3/4aA_2$, which is of the first order, the argument of perigee cannot move around the earth completely, but oscillates around the value 90° as follows:

$$\omega = 90° + \frac{4}{3}\frac{aA_2e_0}{A_3\sin i}\cos\bar{\omega}.$$

The variation of the eccentricity is expressed by

$$e = \frac{3}{4}\frac{A_3}{aA_2}\sin i + e_0\sin\bar{\omega}.$$

In this case $d\Omega_1 = di_1 = 0$, because they have a factor e.

When the inclination is very small, instead of ω, the longitude of perigee, $\pi = \omega + \Omega$ is adopted. The longitude of perigee moves secularly as

$$\pi = \dot{\pi}t + \pi_0,$$

where

$$\dot{\pi} = \frac{A_2}{pa^2}\bar{n}.$$

The variations of $i\sin\Omega$ and $i\cos\Omega$ are derived as in the previous case as

$$i\sin\Omega = i_0\sin\bar{\Omega} + \frac{3}{4}\frac{A_3}{A_2p}e\cos\pi,$$

$$i\cos\Omega = i_0\cos\bar{\Omega} + \frac{3}{4}\frac{A_3}{A_2p}e\sin\pi,$$

where

$$\bar{\Omega} = \dot{\Omega}t + \Omega_0.$$

If i_0 is much smaller than $3eA_3/4A_2p$, then

$$\Omega = \pi + 90° + \frac{4}{3}\frac{A_2pi_0}{A_3e}\cos(\pi - \bar{\Omega}),$$

$$i = \frac{3}{4}\frac{A_3}{A_2p}e + i_0\cos(\pi - \bar{\Omega}).$$

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