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by

M. H. LANE and K. H. CRANFORD
Headquarters Fourteenth Aerospace Force
Ent Air Force Base, Colorado

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M. H. Lane and K. H. Cranford
Deputy of Evaluation
Hq Fourteenth Aerospace Force
Ent Air Force Base, Colorado

Abstract

This paper presents a solution for satellite orbits in a non-rotating atmosphere which takes into account the coupled effects of atmospheric drag and the second, third, and fourth zonal harmonics. It is an extension to the Analytical Drag Theory developed by Lane[3] in 1965. Limitations to the theory for small eccentricities and small inclinations no longer appear. Certain terms with coefficients of the form $B_o{}^2t^2$ and $B_o{}^2t^3$ (where B_o is the drag coefficient and t is the time), which were not present in either Lane's paper or in the Brouwer-Hori[2] theory, were found to be significant for low satellites and have been included. The criterion used in this paper is that terms contributing more than 600 meters in position error over a prediction span of at least ten days (unless decay occurs earlier) would be retained. Where J_2 is the coefficient of the second harmonic in the earth's gravitational potential, the result is complete through second order ($J_2{}^2$) in the secular terms and, except for the long period terms in the mean anomaly, through first order (J_2) in periodic terms. The drag coefficient is assumed to be no larger than J_2.

Introduction

The work presented in this paper was accomplished to improve the capability of the Fourteenth Aerospace Force to perform one of its basic missions (i.e. the cataloging of space objects). It is significant in view of the increasing number of objects in space (currently well over 1700), the amount of computer time required for cataloging, sensor workload, and the large number of objects in the high drag regime.

The analytical theory of the motion of a close earth satellite in an atmosphere has usually been studied by considering separately the effects of atmospheric drag and the asphericity of the earth's gravitational potential. The joint perturbative effects are then deduced by superposition. Strictly speaking, of course, this procedure is not entirely justified for higher order theories. There have been at least two exceptions to this approach. In 1961 Brouwer and Hori[2] published a theory which attempts to take account of the coupled effects of an atmosphere and the earth's oblateness in a single solution. A second paper, by Lane[3], has the general features of the procedure introduced by Brouwer-Hori (B-H), but avoids the two major problems in the B-H theory (discussed below). In both of these papers the method of canonical transformation was used to obtain a description of the motion of an artificial earth satellite under the combined influences of gravity and atmospheric drag.

When the drag-free problem is extended to include the dissipative effects of an atmosphere, the equations of motion take on a different form. The von Zeipel method, used by Brouwer[1] in the drag-free problem, is no longer applicable to this system of differential equations. However, B-H developed a method of transforming such systems of equations canonically, which enabled B-H and Lane to use Brouwer's solution of the problem of artificial satellite theory without drag as a point of departure.

In order to obtain an analytical solution for the system of differential equations with atmospheric drag perturbations included, it is necessary to assume an analytical representation for the atmospheric density. B-H assumed an exponential density function. In order to carry out the integration, B-H expanded this exponential density function in a series complete to the fifth power of the ratio of the eccentricity to the density scale height. As they have indicated this truncation procedure introduces difficulties for satellites which have low perigee heights. Lane showed that if one uses a power density function with an integral exponent the equations of motion may be integrated without expanding the density function in a series, thereby avoiding the associated convergence problem for low perigee heights which appears in the B-H theory. The power density function also appears to provide a better representation of atmospheric density as a function of altitude. This is due to the exponential density function being a special case of the power density function (i.e. when the density scale height is assumed to be invariant with altitude) and the power density function (which assumes a linear variation of scale height with altitude) having one more disposable fitting parameter than the exponential function. Therefore, in Lane's paper the non-rotating spherical exponential model atmosphere of the B-H theory is replaced by a non-rotating spherical power function model of the atmosphere.

In order to write out the explicit form of the system of differential equations, B-H and Lane had to define the explicit relation of the transformed variables to the Delaunay variables. In both papers the relationship between the transformed variables and the Delaunay variables was chosen (one has the freedom to select this relationship at this point in the development) to be that developed between the Delaunay variables and the doubly primed Delaunay variables in Brouwer's drag-free solution. This, incidentally, is how the von Zeipel method enters into the solution of the drag problem.

After some manipulation, B-H and Lane obtained equations of motion in an explicit form and then expressed them in terms of the transformed variables and performed the integration. In the Lane paper, however, in addition to the introduction of the power density function, the introduction of an angle variable (similar to the true anomaly but defined in terms of one parameter of the power density function) was required to effect a significant simplification in the analytical integration process.

In addition to avoiding the problem with low perigee satellites which appears in the B-H theory, Lane also found that the B-H expression,

$$\delta \ell = \delta_s \ell + \delta_{\ell g} \ell + \delta_{sh} \ell$$

should be

$$\delta \ell = \delta_{sh} \ell + \delta_{\ell g} \ell .$$

Setting $\delta_s \ell = 0$ eliminates approximately one-sixth of the terms appearing in the B-H final equations. In a discussion between Brouwer, Hori, and Lane in Spring 1965, both Brouwer and Hori agreed that the $\delta_s \ell$ quantity should be replaced by zero.

In spite of the two problems in the B-H theory discussed above, the authors consider the B-H theory to be a "masterpiece" and would like to reiterate that the procedures and techniques they introduced form the basis for the 1965 Lane paper as well as the extension to that paper presented below.

In the Lane paper the result is complete through second order (J_2^2) in the secular terms and, except for the long-period terms in the mean anomaly, through first order (J_2) in periodic terms. The drag coefficient B_0 (defined in Eqs. 6) is assumed to be no larger than J_2 but is typically much less than J_2. For the long period terms in the mean anomaly, the result is complete through order B_0/J_2 where all terms are complete to the fourth power of the eccentricity. Limitations to Lane's theory for (1) small eccentricities, (2) low inclinations and (3) inclinations near the critical angle are the same as those which appear in the B-H development.

Since approximately ninety percent of the artificial earth satellites have an eccentricity less than 0.1, the "small eccentricity problem" which appears in the Lane paper limits its use for the cataloging of space objects. In 1963 Lyddane[4] developed a reformulation of Brouwer's drag-free equations in order to avoid the "small eccentricity problem." Therefore, since Brouwer's drag-free equations give the solution of the Delaunay variables in terms of the transformed, doubly primed, variables in Lane's equations, Lyddane's method is used to avoid the "small eccentricity problem" in this part of the equations. The remainder of the equations, namely the solution for the doubly primed variables (i.e. the drag portion of the equations), still contain the "small eccentricity problem." In the present paper an approach, which is in some respects analogous to Lyddane's, is developed and used to reformulate the drag portion of Lane's equations in order to avoid the "small eccentricity" and "small inclination" problems. Additional significant terms which have coefficients of the form $B_0^2 t^2$ and $B_0^2 t^3$ are also developed and included in the resulting solution.

The Drag-Free Small Eccentricity and Inclination Problem

Since Brouwer's drag-free equations were used in the development of Lane's drag equations and to transform the coupled drag perturbed, doubly primed Delaunay variables, to the combined drag and gravitationally perturbed unprimed Delaunay variables, a review of the small eccentricity and inclination problem in the drag-free case is useful in justifying the approach taken for avoiding the same problems in the drag case.

In developing his drag-free theory Brouwer started with the equations of motion in the Hamilton canonical form, expressed in terms of the Delaunay variables L, G, H, ℓ, g, h:

$$\frac{dL_j}{dt} = \frac{\partial F}{\partial \ell_j}, \quad \frac{d\ell_j}{dt} = -\frac{\partial F}{\partial L_j}, \qquad F = F(L, G, H, \ell, g, -), \tag{1}$$

where L_j, ℓ_j ($j = 1, 2, 3$) are used for the Delaunay variables. In this system the Hamiltonian, F, is free of the angle variable h. Brouwer then used two successive von Zeipel "canonical" transformations to eliminate the dependence of the Hamiltonian on the remaining two angle variables, ℓ and g. Thus, he arrived at a set of doubly primed Delaunay variables L", G", H", ℓ", g", h" which satisfy the equations

$$\frac{dL_j''}{dt} = 0, \quad \frac{d\ell_j''}{dt} = -\frac{\partial F^{**}}{\partial L_j''}, \qquad F^{**} = F^{**}(L'', G'', H''), \tag{2}$$

where L_j'', ℓ_j'' ($j = 1, 2, 3$) are used for the doubly primed Delaunay variables. Equations (2) are immediately integrable. The Delaunay variables are then expressible, through the von Zeipel algebraic transformations, in terms of these integrals. Hence, the solution is given in terms of the initial conditions and the time. The von Zeipel algebraic transformations, relating L_j'', ℓ_j'' to L_j, ℓ_j, contain first and second order periodic terms with coefficients of the form 1/e and 1/sin I which are undefined for zero eccentricity and inclination constants. Rigorously, the von Zeipel algebraic transformations developed by Brouwer are of the form

$$L_j = L_j\,(L'', G'', H'', \ell, \ell'', g, g', g''), \qquad \ell_j = \ell_j\,(L'', G'', H'', \ell, \ell'', g, g', g'', h'') \tag{3}$$

In order to avoid the problem of solving a system of transcendental equations, after the first transformation Brouwer assumed $\ell = \ell'$, and $g = g'$, and after the second (last) transformation he assumed $g' = g''$ obtaining equations of the form

$$L_j = L_j\,(L'', G'', H'', \ell'', g''), \qquad \ell_j = \ell_j\,(L'', G'', H'', \ell'', g'', h'') \tag{4}$$

Lyddane avoided the zero eccentricity and inclination constant problems as well as the solution of a system of transcendental equations (for small $e \neq 0$) by reformulating Brouwer's perturbation theory in terms of Poincare variables rather than Delaunay variables and calculating from the results, nonsingular osculating elements of the perturbed orbit. Lyddane obtained the following computational formulas:

$$a = a'' + \delta a, \tag{5}$$

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$$\ell+g+h = \ell''+g''+h''+\delta(\ell+g+h),$$

$$e \cos \ell = (e''+\delta e) \cos \ell''-e'' \delta\ell \sin \ell'',$$

$$e \sin \ell = (e''+\delta e) \sin \ell''+e'' \delta\ell \cos \ell'',$$

$$\sin (I/2) \cos h = [\sin (I''/2) + \cos (I''/2)(1/2)\delta I] \cos h'' - \sin (I''/2)\, \delta h \sin h'',$$

$$\sin (I/2) \sin h = [\sin (I''/2) + \cos (I''/2)(1/2)\delta I] \sin h'' + \sin (I''/2)\, \delta h \cos h'',$$

where the δ quantities are those obtained from the Brouwer theory with one important difference. Brouwer's δ quantities are functions of ℓ, g, and g' rather than functions of ℓ'', and g'' (i.e. $\delta\ell_B = \delta\ell_B(L'', G'', H'', \ell, g, g')$), where the subscript B on $\delta\ell$ denotes Brouwer's $\delta\ell$. The δ quantities obtained by Lyddane in equation (5) are the same as Brouwer's, with ℓ replaced by ℓ'' and g and g' replaced by g'' (i.e. $\delta_L\ell = \delta\ell_L(L'', G'', H'', \ell'', g'')$). Lyddane essentially acknowledged this difference when he stated that ℓ'' and g'' must be substituted for Brouwer's ℓ' and g' (note: This takes care of replacing ℓ and g by ℓ'' and g'' since Brouwer after his first von Zeipel transformation had already assumed $\ell = \ell'$ and $g = g'$). An attempt was made to obtain, by a Taylor series expansion, equations identical to those of Lyddane [Eqs. (5)]. This approach could not be justified since the resultant δ quantities (Brouwer's) are functions of ℓ, g, and g' whereas Lyddane's δ quantities are functions of ℓ'' and g''. Furthermore, the dropping of second order singular expansion terms could not be justified.

The Drag Small Eccentricity and Inclination Problem

When the drag-free problem was extended to include the dissipative effects of an atmosphere, Lane obtained equations of motion of the form

$$\frac{dL_j''}{dt} = - BV \frac{1}{(r-s)^\tau} p_j - BV \frac{1}{(r-s)^\tau} \delta p_j,$$
$$\frac{d\ell_j''}{dt} = - \left(\frac{\partial F^{**}}{\partial L_j''}\right) + BV \frac{1}{(r-s)^\tau} q_j + BV \frac{1}{(r-s)^\tau} \delta q_j, \tag{6}$$

where

$$B \equiv B^*(q_o-s)^\tau, \quad B^* \equiv 1/2\, C_D \frac{A'}{m} \rho_o,$$

$$B_o \equiv B(a-s)^{-\tau}$$

and B_o is assumed to be no larger than the first order quantity J_2. Here C_D is the aerodynamic drag coefficient, A' is the effective cross-sectional area of a satellite of mass m, and ρ_o is the atmospheric density at the radial reference height q_o. The quantities τ and s are disposable (i.e. fitting) parameters of the density function and the definitions of the p_j and q_j quantities are:

$$p_1 = L_1[(2a/r)-1], \quad q_1 = 2e\sin E + (2/e)(L_2/L_1) \sin f$$
$$p_2 = L_2, \qquad q_2 = - (2/e) \sin f, \tag{7}$$
$$p_3 = L_3, \qquad q_3 = 0.$$

The δp_j and δq_j quantities are obtained from the Brouwer drag-free theory and appear in Eqs. (6) due to the relationship between the unprimed and doubly primed Delaunay variables, being defined as that developed by Brouwer. The q_j and δq_j quantities, obtained from Brouwer's drag-free theory, have the singularities for zero eccentricity and inclination constants. Consequently the small eccentricity and inclination problems exist in Lane's drag theory.

The right sides of Eqs. (6), which are functions of both Delaunay variables and doubly primed variables, must be expressed completely in terms of doubly primed variables before integration. After some manipulation, Lane expressed these equations explicitly in terms of the doubly primed variables and used the method of successive approximations (the von Zeipel method cannot be used in this non-conservative system) to integrate the resulting system of equations. Analogous to the drag-free case, the equations for ℓ'', g'', and h'' contain periodic terms with coefficients of the form 1/e'' and 1/sin I''.

One cannot revert to Poincare variables to avoid the singularity problem, as Lyddane did in the drag-free case, without completely redeveloping both the drag-free and drag theories (i.e. Lyddane did not obtain an explicit solution for the drag-free problem in terms of the Poincare variables). Even if this were attempted, it is questionable as to whether or not one could then, analytically, integrate the new equations of motion for the drag case.

One of the requirements for using Lyddane's rigorous approach to eliminate the small eccentricity and inclination singularities is that the dynamical system under consideration be conservative. Since the drag problem is dissipative, use of a Lyddane-type approach is difficult, if not impossible, to justify with mathematical rigor. Furthermore, just as one cannot justify obtaining Lyddane's drag-free equations by Taylor series expansions, one cannot justify similar equations in the drag case by such expansions. Since no rigorous approach was obvious, it was decided to develop equations analogous to Lyddane's for the drag case, and rely on numerical results to verify the procedure.

In the drag-free case, the singularity problems come in only through the quantities $\delta\ell_j = \ell_j(t)-\ell_j''(t)$, which are purely periodic functions which contain first and second order terms with 1/e and 1/sin I as coefficients. In the drag case, an analog of $\delta\ell_j$ is $\delta\ell_j'' = \ell_j''(t)-\ell_{js}''(t)$, where $\ell_{js}''(t)$ contains all of the secular components of $\ell_j''(t)$ and the $\delta\ell_j''$ quantities are purely periodic functions which contain first and second order terms with 1/e'' and 1/sin I'' as coefficients. Since the small e'' and I'' problems also occur only in the $\delta\ell_j''$ in the drag case, it was assumed that Lyddane's equations with $\ell_j''(t)$ and $\delta\ell_j$ replaced by $\ell_{js}''(t)$ and $\delta\ell_j''$, respectively, were valid for the drag case. A system of equations, for the drag problem, analogous to Lyddane's can be written as follows:

$$L'' = L''(t_o) + \delta L'',$$

$$\ell''+g''+h'' = \ell_s''+g_s''+h_s''+\delta(\ell''+g''+h''),$$

$$e'' \cos \ell'' = [e''(t_o)+\delta e''] \cos \ell_s''-e''(t_o)\delta\ell'' \sin \ell_s'',$$

$$e'' \sin \ell'' = [e''(t_o)+\delta e''] \sin \ell_s''+e''(t_o)\delta\ell'' \cos \ell_s'',$$

$$\sin (I''/2) \cos h'' = [\sin(I''(t_o)/2) + \cos(I''(t_o)/2)(1/2)\delta I''] \cos h_s'' - \sin(I''(t_o)/2)\delta h'' \sin h_s'',$$

$$\sin (I''/2) \sin h'' = [\sin(I''(t_o)/2) + \cos(I''(t_o)/2)(1/2)\delta I''] \sin h_s'' + \sin(I''(t_o)/2)\delta h'' \cos h_s'', \quad (8)$$

where

$$\delta e'' = e''(t) - e''(t_o),$$

$$\delta I'' = I''(t) - I''(t_o),$$

and $\delta\ell_j'' = \ell_j''(t) - \ell_{js}''(t)$.

This system of equations (Eqs. 8), where the δ quantities are obtained from Lane's equations, along with Eqs. 5 are used to obtain a solution for satellite orbits which takes into account the coupled effects of atmospheric drag and the second, third, and fourth zonal harmonics. Limitations for small eccentricities and small inclinations no longer appear.

Higher Order Terms

In the process of checking the original Lane equations to insure that all terms which could introduce a 600 meter position error (over a prediction span of at least ten days or until decay) were included, it was found that two types of higher order terms should be added. These terms were also absent in the B-H development. The first group of terms arise in the transformation of the drag perturbed differential equations (Eqs. 6) from unprimed to doubly primed variables. To carry out this transformation, it is necessary to expand the terms containing p_j and q_j (which are expressed in terms of the unprimed variables) in Taylor series about the doubly primed variables. Both B-H and Lane carried only the zeroth and first derivative terms in these expansions. Lane used the expression

$$\psi = \psi'' + \left(\frac{\partial\psi}{\partial a}\right)'' \delta a + \left(\frac{\partial\psi}{\partial e}\right)'' \delta e + \left(\frac{\partial\psi}{\partial y}\right)'' \delta y + \left(\frac{\partial\psi}{\partial I}\right)'' \delta I + \left(\frac{\partial\psi}{\partial E}\right)'' \delta E,$$

where ψ represents any one of the ψp_j, ψq_j, and

$$\psi p_j = - BV \frac{1}{(r-s)^\tau} p_j,$$

$$\psi q_j = + BV \frac{1}{(r-s)^\tau} q_j.$$

In the present development, however, it was found that the second derivative terms

$$\left[\frac{1}{2}\left(\frac{\partial^2\psi}{\partial a^2}\right)'' \delta a^2 + \left(\frac{\partial^2\psi}{\partial a \partial y}\right)'' \delta a\delta y + \frac{1}{2}\left(\frac{\partial^2\psi}{\partial y^2}\right)'' \delta y^2\right]$$

in these expansions were significant for satellites with very low perigees. The additional terms in the integrated equations introduced by these extended expansions are given in the appendix.

The method of successive approximations is used to integrate the differential equations in terms of the doubly primed variables in both the Lane paper and the B-H paper. They used the drag-free variables as the zeroth approximation, integrated the differential equations for L'', G'', and H'' once, substituted the result of this first approximation into the dominant term in the ℓ'' equation (this is the only zeroth order term in the differential equations), and integrated this term. In the present development, the integration is carried one step further. The secular parts of the first integration for L'', G'', and H'' are used as the first order approximations, and the L'', G'', and H'' differential equations are reintegrated. The result of this integration is used in the integration of the dominant term in the ℓ'' equation. The terms resulting from this extended integration procedure comprise the second group of higher order terms, and are listed in the Appendix.

The omission of these higher order terms can contribute errors of the order of sixty kilometers to position predictions for satellites with perigees below 200 kilometers.

Comparison With the Results of Numerical Integration

As stated earlier, the development to avoid the small eccentricity and small inclination singularities is difficult, if not impossible, to justify with mathematical rigor. In order to test the theory, satellite observations were generated for six test cases using a numerical integration program which used the Jacchia[5] model atmosphere and the geopotential coefficients J_2, J_3, J_4, and J_5. Each of the test cases had an inclination of 66.69°, argument of perigee of 100°, longitude of ascending node of 75°, and $1/2\ C_D \frac{A}{m}$ equal to 0.001 m^2/kg. The eccentricities and perigee heights for the test cases are given in the table below.

ORBITAL CHARACTERISTICS TABLE

Test Case #	Eccentricity	Perigee Height (km)
1	0.00001	200
2	0.001	200
3	0.1	200
4	0.00001	500
5	0.001	500
6	0.1	500

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Observations were generated at 20 minute intervals for a period of two weeks for the test cases with 200 km perigee height, and at 40 minute intervals for a period of four weeks for the test cases with 500 km perigee height. The testing procedure used was to perform a differential correction using the first one-third of the observations and then make a prediction and comparison with the remaining observations.

The test results showed that periodic variations attributable to small eccentricities for drag perturbed satellites are comparable to those in the drag-free case where the Lyddane modification has been used. This indicates that the zero eccentricity singularities have been removed in the Gravitational-Drag coupled theory. Additional test cases have not yet been generated to verify that the zero inclination singularities have been removed. For actual satellites, this singularity is far less common than the zero eccentricity singularity. In view of the marked similarity of the approaches taken to avoid the zero eccentricity and inclination singularities, the authors are confident that the zero inclination singularities have also been removed.

Summary

The reformulated version of Lane's Equations is a solution for satellites in a non-rotating atmosphere which takes into account the coupled effects of atmospheric drag and the second, third, and fourth zonal harmonics. Limitations to the theory for small eccentricities and small inclinations no longer appear. Where J_2 is the coefficient of the second harmonic in the earth's gravitational potential, the result is complete through second order (J_2^2) in the secular terms and, except for the long period terms in the mean anomaly, through first order (J_2) in periodic terms. The drag coefficient is assumed to be no larger than J_2. Terms contributing more than 600 meters in position error over a prediction span of at least ten days (unless decay occurs earlier) are retained.

The form in which the present theory is developed lends itself to long term position predictions, efficient use of computer time, satellite decay predictions and extensions which could take account of higher order neglected perturbations such as the diurnal effect. One may account, to a certain degree, for the diurnal effect in the manner described in Lane's paper.

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Glossary of Symbols

r = distance from geocenter to satellite

ρ_0 = atmospheric density at q_0

q_0 = reference height, taken to be 120 km.

s = fitting parameter in density representation

ι = fitting parameter in density representation

$\rho = \rho_0\left(\frac{q_0-s}{r-s}\right)^\iota$ = assumed density representation

$\eta = \frac{ae}{a-s}$, where a is the semimajor axis of the orbit and e its eccentricity

$\cos\lambda = \frac{\cos E-\eta}{1-\eta\cos E}$, where E is the eccentric anomaly

$\sin\lambda = \frac{\sqrt{1-\eta^2}\sin E}{1-\eta\cos E}$

$\gamma_1 = \eta + \cos\lambda$

$\gamma_2 = 1 + \eta\cos\lambda$

$\beta = \sqrt{1-e^2}$

$\xi = (a-s)^{-1}$

$\alpha = (1-\eta^2)^{1/2-\iota}\xi^\iota$

C_D = Aerodynamic drag coefficient

A' = effective cross-sectional area of satellite

m = mass of satellite

$B_0 = \frac{1}{2} C_D \frac{A'}{m} \rho_0 \left(\frac{q_0-s}{a-s}\right)^\iota \lesssim J_2$

$B = B_0(a-s)^\iota$

$\theta = \cos I$, I = inclination

n = mean motion

k_2 = second zonal coefficient in geopotential ($k_2 = J_2R^2/2$, where R is the earth's equatorial radius)

$A_{3,0}$ = third zonal coefficient in geopotential ($A_{3,0} = -\mu J_3R^3$)

k_4 = fourth zonal coefficient in geopotential ($k_4 = -3J_4R^4/8$)

$A_{5,0}$ = fifth zonal coefficient in geopotential $(A_{5,0} = -\mu J_5 R^5)$

$$I_{j,k} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos j x}{(1-\eta\cos x)^k}\, dx$$

$$J_n = \frac{1}{2\pi}\int_0^{2\pi} (1+\eta\cos x)^n\, dx$$

$$y = -e\cos E$$

Appendix

The equations used for computing the osculating elements L, G, H, ℓ, g, h are given in the sequence of steps below. It is understood that (unless other wise noted) all the variables in the equations in steps (1) through (6) are doubly primed, with L_i'' on the right hand side evaluated at epoch and ℓ_i'' on the right hand side taken to be ℓ_{is}''. It is assumed that epoch elements L_o'', G_o'', H_o'', ℓ_o'', g_o'', h_o'' and epoch time t_o, along with time t, are given. It is further assumed that τ is an integer greater than or equal to 3, that $s<q_o$, and that $q_o<a$.

(1) ℓ_s'', g_s'', h_s'' are computed iteratively. Starting values used are $\ell_{D/F}''$, $g_{D/F}''$, $h_{D/F}''$. Equations are:

$$\ell_{D/F}'' = \ell_o'' + \{1 + \frac{3}{2}\frac{k_2}{a^2\beta^3}(-1+3\theta^2) + \frac{3}{32}\frac{k_2^2}{a^4\beta^7}[-15 + 16\beta + 25\beta^2 + (30-96\beta-90\beta^2)\theta^2 + (105+144\beta + 25\beta^2)\theta^4] + \frac{15}{16}\frac{k_4}{a^4\beta^7}e^2(3-30\theta^2+35\theta^4)\}n(t-t_o),$$

$$g_{D/F}'' = g_o'' + \{-\frac{3}{2}\frac{k_2}{a^2\beta^4}(1-5\theta^2) + \frac{3}{32}\frac{k_2^2}{a^4\beta^8}[-35+24\beta + 25\beta^2 + (90-192\beta-126\beta^2)\theta^2 + (385+360\beta+45\beta^2)\theta^4] + \frac{5}{16}\frac{k_4}{a^4\beta^8}[21-9\beta^2 + (-270+126\beta^2)\theta^2 + (385 - 189\beta^2)\theta^4]\}n(t-t_o),$$

$$h_{D/F}'' = h_o'' + \{-3\frac{k_2}{a^2\beta^4}\theta + \frac{3}{8}\frac{k_2^2}{a^4\beta^8}[(-5+12\beta+9\beta^2)\theta + (-35-36\beta-5\beta^2)\theta^3] + \frac{5}{4}\frac{k_4}{a^4\beta^8}\theta(3-7\theta^2)(5-3\beta^2)\}n(t-t_o),$$

$$\ell_s'' = \ell_{D/F}'' - 3n[1 + \frac{2k_2}{a^2\beta^3}(-1+3\theta^2)]\int_{t_o}^{t}\Delta L\, dt - \frac{9}{2}n\frac{k_2}{a^2\beta^3}(-1+5\theta^2)\int_{t_o}^{t}\Delta G\, dt + 9n\frac{k_2}{a^2\beta^3}\theta^2\int_{t_o}^{t}\Delta H\, dt,$$

$$g_s'' = g_{D/F}'' + \frac{9}{2}n\frac{k_2}{a^2\beta^4}(1-5\theta^2)\int_{t_o}^{t}\Delta L dt + 3n\frac{k_2}{a^2\beta^4}(2-15\theta^2)\int_{t_o}^{t}\Delta G dt + 15n\frac{k_2}{a^2\beta^4}\theta^2\int_{t_o}^{t}\Delta H dt,$$

$$h_s'' = h_{D/F}'' + 9n\frac{k_2\theta}{a^2\beta^4}\int_{t_o}^{t}\Delta L dt + 15n\frac{k_2\theta}{a^2\beta^4}\int_{t_o}^{t}\Delta G dt - 3n\frac{k_2\theta}{a^2\beta^4}\int_{t_o}^{t}\Delta H dt,$$

where

$$\int_{t_o}^{t}\Delta L dt = -B\alpha a[1 - \frac{3k_2}{a^2}\left(\frac{L_o''}{G_o''}\right)^3(-\frac{1}{2}+\frac{3}{2}\theta^2)]\, I_1$$

$$- Bk_2\xi^\tau(-\frac{1}{2}+\frac{3}{2}\theta^2)\frac{1}{2n}[1 - \frac{3k_2}{a^2}\left(\frac{L_o''}{G_o''}\right)^3(-\frac{1}{2} + \frac{3}{2}\theta^2)]\,(E-E_o)^2\,\{-\tau\xi[-3I_{o,\tau+1} - \frac{9}{2}e\, I_{1,\tau+1} + e^2(-\frac{9}{2}I_{o,\tau+1} - I_{2,\tau+1}) + e^3(-\frac{97}{16}I_{1,\tau+1} - \frac{9}{16}I_{3,\tau+1})] + \frac{3}{a}[I_{o,\tau} + \frac{11}{2}eI_{1,\tau} + e^2(4I_{o,\tau} + 5I_{2,\tau})]\}$$

$$+ \frac{a^4\beta^8 B\xi^\tau(1-\theta^2)}{3nk_2(1-5\theta^2)^2}\{-\tau\xi[\frac{1}{6}I_{2,\tau+1} + \frac{e}{3}I_{3,\tau+1} + e^2(-\frac{1}{16}I_{o,\tau+1} + \frac{1}{8}I_{2,\tau+1} + \frac{17}{48}I_{4,\tau+1}) + \frac{e^3}{24}(-3I_{1,\tau+1} + 8I_{3,\tau+1} + 7I_{5,\tau+2})] + \frac{1}{a}[I_{2,\tau} + \frac{e}{4}(3I_{1,\tau} + 11I_{3,\tau}) + \frac{e^2}{4}(I_{o,\tau} + 8I_{2,\tau} + 17I_{4,\tau})]\}\,[\cos 2g(t) - \cos 2g(t_o) + 2(g(t) - g(t_o))\sin 2g(t_o)]$$

$$+ \frac{a^4\beta^6 B\xi^\tau}{3nk_2(1-5\theta^2)^2}\{\frac{1}{24}[1-11\theta^2 - \frac{40\theta^4}{1-5\theta^2}] - \frac{5}{36}\frac{k_4}{k_2^2}[1-3\theta^2 - \frac{8\theta^4}{1-5\theta^2}]\}\{-\tau\xi[-eI_{1,\tau+1} + \frac{e^2}{2}(-I_{o,\tau+1} - 3I_{2,\tau+1}) + \frac{e^3}{8}(-3I_{1,\tau+1} - 9I_{3,\tau+1})] + \frac{3}{a}[eI_{1,\tau} + \frac{e^2}{2}(I_{o,\tau} + 3I_{2,\tau})]\}[\cos 2g(t) - \cos 2g(t_o) + 2(g(t)-g(t_o))\sin 2g(t_o)]$$

$$+ \frac{a^5\beta^8 B\xi^\tau A_{3,o}\sin I}{9k_2^3 n(1-5\theta^2)^2}\{-\tau\xi[-I_{1,\tau+1} + \frac{e}{2}(-I_{o,\tau+1}$$

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$$- 3I_{2,\tau+1}) + \frac{e^2}{8} (-3I_{1,\tau+1} - 9I_{3,\tau+1})$$

$$+ \frac{e^3}{16} (3I_{0,\tau+1} - 8I_{2,\tau+1} - 11I_{4,\tau+1})]$$

$$+ \frac{3}{a} [I_{1,\tau} + \frac{e}{2} (I_{0,\tau} + 3I_{2,\tau}) + \frac{e^2}{8} (9I_{1,\tau}$$

$$+ 11I_{3,\tau})]\}[\sin g(t) - \sin g(t_0) - (g(t)$$

$$-g(t_0))\cos g(t_0)] + \delta\int_{t_0}^{t} \Delta L dt$$

$$+ \frac{B\tau}{L''_0}\int_{t_0}^{t} F_1 dt - \frac{2B^2}{3n} [\alpha a + 3k_2\tau\, \xi^{\tau+1} (- \frac{1}{2}$$

$$+ \frac{3}{2}\theta^2)]^2 (E-E_0)^3,$$

$$\int_{t_0}^{t} \Delta G dt = \int_{t_0}^{t} \Delta H dt = - B\alpha a[1 - \frac{3k_2}{a^2}\left(\frac{L''_0}{G''_0}\right)^3 (- \frac{1}{2}$$

$$+ \frac{3}{2}\theta^2)] I_2 - \frac{3Bk_2\xi^\tau}{2n} (- \frac{1}{2} + \frac{3}{2}\theta^2)\tau\, \xi I_{0,\tau+1}(E-E_0)^2$$

$$- \frac{Ba^4\xi^\tau\beta^8}{18nk_2} \frac{(1-\theta^2)\tau\,\xi}{(1-5\theta^2)^2} \{\cos 2g(t) - \cos 2g(t_0) + 2(g(t)$$

$$-g(t_0)) \sin 2g(t_0)\} - \frac{Ba^4\xi^\tau\beta^8\tau\,\xi}{3nk_2(1-5\theta^2)^2}[\frac{e}{\beta^2}\{\frac{1}{24}[1-11\theta^2$$

$$- \frac{40\theta^4}{1-5\theta^2}] - \frac{5}{36}\frac{k_4}{k_2^2} [1-3\theta^2 - \frac{8\theta^4}{(1-5\theta^2)}]\} I_{1,\tau+1}]$$

$$\times\{\cos 2g(t) - \cos 2g(t_0) + 2(g(t)-g(t_0))\sin 2g(t_0)\}$$

$$+ \frac{Ba^5\xi^\tau\beta^8\tau\,\xi A_{3,0}}{9nk_2^3(1-5\theta^2)^2} (\sin I)I_{1,\tau+1} \{\sin g(t)$$

$$-\sin g(t_0) - (g(t)-g(t_0))\cos g(t_0)\},$$

and $$I_1 = \int_{t_0}^{t}\int_{\lambda_0}^{\lambda} (\gamma_2^{\tau-1} + 2e\gamma_1\gamma_2^{\tau-2} + \frac{3}{2}e^2\gamma_1^2\gamma_2^{\tau-3}$$

$$+ e^3\gamma_1^3\gamma_2^{\tau-4} + \frac{7}{8}e^4\gamma_1^4\gamma_2^{\tau-5})d\lambda dt$$

$$= \frac{1}{2n}\{(1 + \frac{2e}{\eta} + \frac{3e^2}{2\eta^2} + \frac{e^3}{\eta^3} + \frac{7e^4}{8\eta^4}) J_{\tau-1}$$

$$- (1-\eta^2)(\frac{2e}{\eta} + \frac{3e^2}{\eta^2} + \frac{3e^3}{\eta^3} + \frac{7e^4}{2\eta^4}) J_{\tau-2}$$

$$+ (1-\eta^2)^2 (\frac{3e^2}{2\eta^2} + \frac{3e^3}{\eta^3} + \frac{21e^4}{4\eta^4}) J_{\tau-3}$$

$$- (1-\eta^2)^3 (\frac{e^3}{\eta^3} + \frac{7e^4}{2\eta^4}) J_{\tau-4}$$

$$+ (1-\eta^2)^4 \frac{7e^4}{8\eta^4} J_{\tau-5}\} (\lambda-\lambda_0)^2,$$

$$I_2 = \int_{t_0}^{t}\int_{\lambda_0}^{\lambda} (\gamma_2^{\tau-1} - \frac{1}{2}e^2\gamma_1^2\gamma_2^{\tau-3})d\lambda dt$$

$$= \frac{1}{2n}\{(1 - \frac{e^2}{2\eta^2}) J_{\tau-1} + \frac{e^2}{2\eta^2} (1-\eta^2)[2 J_{\tau-2}$$

$$- (1-\eta^2) J_{\tau-3}]\} (\lambda-\lambda_0)^2.$$

In the above equation for $\int_{t_0}^{t} \Delta L dt$, $\int_{t_0}^{t} \Delta G dt$, $\int_{t_0}^{t} \Delta H dt$, $g(t)$ stands for $g''_s(t)$ and λ is related to $\ell''_s(t)$ through the equation

$$\cos\lambda = \frac{\cos E - \eta}{1 - \eta\cos E}$$

$$\sin\lambda = \frac{\sqrt{1-\eta^2}\,\sin E}{1 - \eta\cos E}$$

$$\ell''_s(t) = E - e''(t_0)\sin E.$$

The presence of $\ell''_s(t)$ and $g''_s(t)$ in the integrals makes iteration necessary for the solution of $\ell''_s(t)$, $g''_s(t)$, $h''_s(t)$. Also in the above equations,

$$\frac{B\tau}{L''_0}\int_{t_0}^{t} F_1 dt = - \frac{B^2\alpha^2a^3\tau\,\xi}{n(1-\eta^2)} J_\tau J_{\tau-1} \frac{(\lambda-\lambda_0)^3}{3}$$

$$- \frac{B^2k_2\tau^2\xi^{2\tau+2}a^2}{n} (- \frac{1}{2} + \frac{3}{2}\theta^2) I^2_{0,\tau+1} (E-E_0)^3,$$

and

$$\delta\int_{t_0}^{t} \Delta L dt = - \frac{B\tau(\tau+1)\xi^{\tau+2}}{4an} [9k_2^2(- \frac{1}{2} + \frac{3}{2}\theta^2)^2$$

$$+ \frac{a^2A_{3,0}^2\sin^2 I}{32k_2^2} + \frac{k_2^2\sin^4 I}{4}](1-\eta^2)^{\tau+\frac{1}{2}} I_{0,\tau+2}(\lambda-\lambda_0)^2$$

$$- \frac{2B\tau(\tau+1)\beta^8\xi^\tau a^3}{9n(1-5\theta^2)^2}\{- \frac{aA_{3,0}\sin I}{8k_2^2} (\sin^2 I + 12) I_{1,\tau+2}$$

$$\times [\sin g(t) - \sin g(t_0) - (g(t)-g(t_0)) \cos g(t_0)]$$

$$+ \frac{\sin^2 I}{2} [\frac{a^2A_{3,0}^2}{32k_2^4} + \frac{3}{4}(3\theta^2-1)] I_{2,\tau+2}[-\frac{1}{2}(\cos 2g(t)$$

$$- \cos 2g(t_0)) - (g(t)-g(t_0)) \sin 2g(t_0)]$$

$$+ \frac{aA_{3,0}\sin^3 I}{24k_2^2} I_{3,\tau+2} [\frac{1}{3} (\sin 3g(t) - \sin 3g(t_0))$$

$$- (g(t)-g(t_o)) \cos 3g(t_o)] + \frac{\sin^4 I}{16} I_{4,\tau+2}$$

$$\times [-\frac{1}{4} (\cos 4g(t) - \cos 4g(t_o))$$

$$- (g(t)-g(t_o)) \sin 4g(t_o)]\}.$$

The quantities $\frac{B\tau}{L''_o}\int_{t_o}^{t} F_1 dt$ and $\delta\int_{t_o}^{t} \Delta L dt$ arise from the presence of the functions F_1 and F_2 in $L''(t)$. (See step 3.)

(2) Quantities $e\delta\ell$, δI, and $(\sin \frac{I}{2})\delta h$ are computed for later use in step (6). Equations are:

$$e\delta\ell = B\alpha a \sqrt{1-\eta^2} I_3 + Ba^2\xi^\tau \beta^5 \frac{1-\theta^2}{1-5\theta^2} \{-\frac{1}{6}\tau\xi[I_{3,\tau+1}$$

$$- I_{1,\tau+1} + e(\frac{1}{4} I_{o,\tau+1} - \frac{25}{8} I_{2,\tau+1} + \frac{15}{4} I_{4,\tau+1}$$

$$- \frac{7}{8} I_{6,\tau+1})] + \frac{1}{6a}[-3I_{1,\tau} + 7I_{3,\tau}]\} [\cos 2g(t)$$

$$- \cos 2g(t_o)] + \frac{Ba\xi^\tau\beta^3}{1-5\theta^2} \{\frac{1}{24}(1-11\theta^2 - \frac{40\theta^4}{1-5\theta^2})$$

$$- \frac{5}{36}\frac{k_4}{k_2^2}(1-3\theta^2 - \frac{8\theta^4}{1-5\theta^2})\} [-\tau a\xi e(I_{o,\tau+1} - I_{2,\tau+1})$$

$$+ 2I_{1,\tau}][\cos 2g(t) - \cos 2g(t_o)]$$

$$- \frac{Ba^2\xi^\tau\beta^5 A_{3,0} \sin I}{6k_2^2(1-5\theta^2)} \{\tau a\xi[I_{o,\tau+1} - I_{2,\tau+1}$$

$$+ e(I_{1,\tau+1} - I_{3,\tau+1})]$$

$$+ 3(I_{o,\tau} - I_{2,\tau})\} [\sin g(t) - \sin g(t_o)],$$

$$\delta I = \frac{Ba\xi^\tau\beta^4\theta}{1-5\theta^2} \{\frac{2}{3} I_{2,\tau} \sin I [\sin 2g(t) - \sin 2g(t_o)]$$

$$- \frac{aA_{3,0}}{3k_2^2} I_{1,\tau} [\cos g(t) - \cos g(t_o)]\},$$

$$(\sin \frac{I}{2})\delta h = -\frac{2}{3}\frac{Ba\xi^\tau\theta}{1-5\theta^2}(\sin \frac{I}{2}) I_{2,\tau} [\cos 2g(t)$$

$$- \cos 2g(t_o)] - \frac{Ba^2\xi^\tau\beta^2 A_{3,0}\theta}{6k_2^2(1-5\theta^2)\cos\frac{I}{2}} [I_{1,\tau} + \frac{e}{2}(-I_{o,\tau}$$

$$+ I_{2,\tau})] [\sin g(t) - \sin g(t_o)],$$

where $I_3 = \int_{\lambda_o}^{\lambda} \sin\lambda\, (2\gamma_2^{\tau-2} + 2e\gamma_1\gamma_2^{\tau-3})d\lambda$

$$= -\frac{2}{\eta^2}\{\frac{\eta+e}{\tau-1}[(1+\eta\cos\lambda)^{\tau-1} - (1+\eta\cos\lambda_o)^{\tau-1}]$$

$$- \frac{e(1-\eta^2)}{\tau-2}[(1+\eta\cos\lambda)^{\tau-2} - (1+\eta\cos\lambda_o)^{\tau-2}]\}.$$

(3) L'' and H'' are computed as follows:

$$L'' = L''(t_o)[1 - B\alpha a[1 - \frac{3k_2}{a^2}\frac{L''^3_o}{G''^3_o}(-\frac{1}{2} + \frac{3}{2}\theta^2)] \int_{\lambda_o}^{\lambda} (\gamma_2^{\tau-1}$$

$$+ 2e\gamma_1\gamma_2^{\tau-2} + \frac{3}{2}e^2\gamma_1^2\gamma_2^{\tau-3})d\lambda - Bk_2\xi^\tau(-\frac{1}{2}$$

$$+ \frac{3}{2}\theta^2)(E-E_o)\{-\tau\xi[-3I_{o,\tau+1} - \frac{9}{2} e I_{1,\tau+1}]$$

$$+ \frac{3}{a} I_{o,\tau}\} + Ba^2\xi^\tau\beta^4 \frac{(1-\theta^2)}{(1-5\theta^2)} \{-\tau\xi[\frac{1}{6} I_{2,\tau+1}$$

$$+ \frac{1}{3} eI_{3,\tau+1}] + \frac{1}{a} I_{2,\tau}\}(\sin 2g(t) - \sin 2g(t_o))$$

$$+ \frac{Ba^2\xi^\tau\beta^4}{(1-5\theta^2)}\{\frac{e}{\beta^2}(\frac{1}{24}[1-11\theta^2 - \frac{40\theta^4}{1-5\theta^2}] - \frac{5}{36}\frac{k_4}{k_2^2}[1-3\theta^2$$

$$- \frac{8\theta^4}{1-5\theta^2}])(\sin 2g(t) - \sin 2g(t_o))$$

$$- \frac{1}{6}\frac{A_{3,0}}{k_2^2} a \sin I(\cos g(t) - \cos g(t_o))\}\{-\tau\xi[-I_{1,\tau+1}$$

$$- \frac{1}{2} e(I_{o,\tau+1} + 3I_{2,\tau+1})] + \frac{3}{a} I_{1,\tau}\} + \frac{B\tau}{L''_o} F_1(\lambda,E)$$

$$+ \frac{1}{L''_o} F_2(\lambda,g)],$$

$$H'' = H''(t_o)[1 - B\alpha a[1 - \frac{3k_2}{a^2}\left(\frac{L''_o}{G''_o}\right)^3 (-\frac{1}{2}$$

$$+ \frac{3}{2}\theta^2)] \int_{\lambda_o}^{\lambda} (\gamma_2^{\tau-1} - \frac{1}{2} e^2\gamma_1^2\gamma_2^{\tau-3})d\lambda - Bk_2\xi^\tau(-\frac{1}{2}$$

$$+ \frac{3}{2}\theta^2)(E-E_o)\{-\tau\xi[-3I_{o,\tau+1} + \frac{3}{2} eI_{1,\tau+1}] + \frac{1}{a}[3I_{o,\tau}]\}$$

$$+ \frac{Ba^2\xi^\tau\beta^4(1-\theta^2)}{1-5\theta^2}\{-\tau\xi[\frac{1}{6} I_{2,\tau+1} - \frac{e}{6}(I_{1,\tau+1}$$

$$- I_{3,\tau+1})] + \frac{1}{3a} I_{2,\tau}\}[\sin 2g(t) - \sin 2g(t_o)]$$

$$+ \frac{Ba^2\xi^\tau\beta^4}{1-5\theta^2}\{\frac{e}{\beta^2}(\frac{1}{24}[1-11\theta^2 - \frac{40\theta^4}{1-5\theta^2}] - \frac{5}{36}\frac{k_4}{k_2^2}[1-3\theta^2$$

$$- \frac{8\theta^4}{1-5\theta^2}])(\sin 2g(t) - \sin 2g(t_o))$$

$$- \frac{aA_{3,0}}{6k_2^2} \sin I \ (\cos g(t)-\cos g(t_o))\}\{-\iota\xi[-I_{1,\iota+1} + \frac{e}{2}(I_{o,\iota+1} - I_{2,\iota+1})] + \frac{1}{a} I_{1,\iota}\}$$

$$+ \frac{B\iota}{L''_o} F_1(\lambda,E) + \frac{1}{L''_o} F_2(\lambda,g)],$$

where for p and m nonnegative integers,

$$\int^{\lambda} \gamma_1^{p}\gamma_2^{m} d\lambda = \sum_{k=0}^{p}\sum_{\ell=0}^{m} \eta^{p-k+\ell} \binom{p}{k}\binom{m}{\ell}$$

$$\times\{\frac{1}{k+\ell}\cos^{k+\ell-1}\lambda\sin\lambda$$

$$+ \sum_{j=2}^{i-1} \frac{\prod_{k=2}^{j}(\ell-k+3)}{\prod_{k=1}^{j}(\ell-k+2)} \cos^{k+\ell-2j+1}\lambda\sin\lambda \ + R_i\},$$

$$\text{where } i = \begin{cases} \frac{k+\ell+1}{2} \text{ if } k+\ell \text{ is odd} \\ \frac{k+\ell+2}{2} \text{ if } k+\ell \text{ is even} \end{cases}$$

$$\text{and } R_i = \begin{cases} \lambda \text{ if } k+\ell=0 \\ 0 \text{ if } k+\ell=1 \\ \lambda\prod_{k=2}^{i} \left(\frac{2k-3}{2k-2}\right) \text{ if } k+\ell \text{ is even and } k+\ell\geq 2 \\ \sin\lambda\prod_{k=2}^{i}\left(\frac{2k-2}{2k-1}\right) \text{ if } k+\ell \text{ is odd and } k+\ell\geq 3, \end{cases}$$

Also, $$\frac{B\iota}{L''_o} F_1(\lambda,E) = \frac{-2B^2\alpha^2a^3\iota\xi}{1-\eta^2}\int_{\lambda_o}^{\lambda}\gamma_2^{\iota}\ \{\int_{\lambda_o}^{\lambda}\gamma_2^{\iota-1}d\lambda\}d\lambda$$

$$- 6B^2k_2\iota^2\xi^{2\iota+2}a^2(-\frac{1}{2} + \frac{3}{2}\theta^2)(I_{o,\iota+1})\int_{E_o}^{E}\frac{(E-E_o)}{(1-\eta\cos E)^{\iota+1}}dE$$

$$\cong -\frac{B^2\alpha^2a^3\iota\xi}{1-\eta^2} J_\iota \ J_{\iota-1} \ (\lambda-\lambda_o)^2$$

$$- 3B^2k_2\iota^2\xi^{2\iota+2} \ a^2(-\frac{1}{2}+\frac{3}{2}\theta^2) \ I^2_{o,\iota+1} \ (E-E_o)^2,$$

and

$$\frac{1}{L''_o} F_2(\lambda,g) = -\frac{B\iota(\iota+1)\xi^{\iota+2}}{2a} [9k_2^2 \ (-\frac{1}{2}+\frac{3}{2}\theta^2)^2$$

$$+ \frac{a^2A^2_{3,0}\sin^2 I}{32k_2^2} + \frac{k_2^2\sin^4 I}{4}]\int_{\lambda_o}^{\lambda}\gamma_2^{\iota+1}d\lambda$$

$$+ \frac{B\iota(\iota+1)\beta^4\xi^{\iota+2}a}{3(1-5\theta^2)}\{-\frac{aA_{3,0}\sin I}{8k_2}(\sin^2 I+12) \ I_{1,\iota+2}$$

$$\times [\cos g(t)-\cos g(t_o)] + \frac{\sin^2 I}{2}[\frac{a^2A^2_{3,0}}{32k_2^3}$$

$$+ \frac{3k_2}{4}(3\theta^2-1)]I_{2,\iota+2}[\sin 2g(t)-\sin 2g(t_o)]$$

$$+ \frac{aA_{3,0}\sin^3 I}{24k_2} I_{3,\iota+2}[\cos 3g(t)-\cos 3g(t_o)]$$

$$+ \frac{k_2\sin^4 I}{16} I_{4,\iota+2}[\sin 4g(t)-\sin 4g(t_o)]\}.$$

The function F_2 is due to the second derivative terms in the ψ_{p_i} expansion, and F_1 is due to the extended integration procedure.

(4) $\ell''+g''+h''$ is computed. The equation is:

$$\ell''+g''+h'' = \ell''_{D/F}+g''_{D/F}+h''_{D/F}+n\{\frac{3k_2}{a^2\beta^4}[2\beta(1-3\theta^2)$$

$$+ \frac{3}{2}(1-5\theta^2)+3\theta]-3\}\int_{t_o}^{t}\Delta L dt + \frac{nk_2}{a^2\beta^4}[\beta(\frac{9}{2}-\frac{45}{2}\theta^2)$$

$$+ 6+15\theta-45\theta^2]\int_{t_o}^{t}\Delta G dt + \frac{nk_2\theta}{a^2\beta^4}[-3+(15$$

$$+ 9\beta)\theta]\int_{t_o}^{t}\Delta H dt + B\alpha a\sqrt{1-\eta^2}[1-\frac{3k_2}{a^2}\left(\frac{L''_o}{G''_o}\right)^3(-\frac{1}{2}$$

$$+ \frac{3}{2}\theta^2)]e \ I_4 + Ba^2\xi^{\iota} \ \frac{1-\theta^2}{1-5\theta^2}\{-\frac{\iota\xi e}{12}(I_{3,\iota+1}-I_{1,\iota+1})$$

$$+ \frac{1}{6a}[e(-\frac{3}{2}I_{1,\iota}+\frac{7}{2}I_{3,\iota})-3I_{o,\iota}-3I_{2,\iota}]\}$$

$$\times [\cos 2g(t)-\cos 2g(t_o)] - \frac{Ba\xi^{\iota}}{3(1-5\theta^2)} I_{2,\iota}(3+2\theta$$

$$- 5\theta^2)[\cos 2g(t)-\cos 2g(t_o)] + \frac{Ba\xi^{\iota}}{1-5\theta^2}[\frac{1}{24}(1-11\theta^2$$

$$- \frac{40\theta^4}{1-5\theta^2}) - \frac{5}{36}\frac{k_4}{k_2^2}(1-3\theta^2-\frac{8\theta^4}{1-5\theta^2})]eI_{1,\iota}[\cos 2g(t)$$

$$- \cos 2g(t_o)] + \frac{Ba^2\xi^{\iota}A_{3,0}\sin I}{6k_2^2(1-5\theta^2)}\{-\iota a\xi\frac{e}{2}(I_{o,\iota+1}$$

$$- I_{2,\iota+1})-2I_{1,\iota}\}[\sin g(t)-\sin g(t_o)]$$

$$- \frac{Ba^2\xi^{\iota}\beta^2A_{3,0}\theta}{3(1-5\theta^2)k_2^2}\tan(\frac{I}{2})[I_{1,\iota}+\frac{e}{2}(-I_{o,\iota}+I_{2,\iota})]$$

$$\times [\sin g(t)-\sin g(t_o)],$$

where $I_4 = \int_{\lambda_0}^{\lambda} (\sin\lambda)\ \gamma_2^{\tau-2} d\lambda$

$$= -\frac{1}{\eta(\tau-1)}[(1+\eta\cos\lambda)^{\tau-1} - (1+\eta\cos\lambda_0)^{\tau-1}].$$

(5) e'' and G'' are computed, and Δe is computed for use in step (6). Equations are:

$$e'' = e - B\alpha a\beta^2 \int_{\lambda_0}^{\lambda} [2\gamma_1\gamma_2^{\tau-2} + 2e\gamma_1^{2}\gamma_2^{\tau-3}]d\lambda$$

$$- B\beta^2\xi^\tau k_2 \left(-\frac{1}{2} + \frac{3}{2}\theta^2\right)(E-E_0)\{\tau\xi[6I_{1,\tau+1}$$

$$+ \frac{3}{2}e(I_{0,\tau+1} + I_{2,\tau+1})] + \frac{15}{a} I_{1,\tau}\}$$

$$+ \frac{1}{2} B\beta^6 a^2 \xi^\tau \frac{1-\theta^2}{1-5\theta^2}\{\tau\xi[-\frac{4}{3} I_{1,\tau+1} - \frac{2}{3} I_{3,\tau+1}$$

$$+ e(\frac{2}{3} I_{0,\tau+1} + \frac{1}{3} I_{2,\tau+1} - \frac{1}{3} I_{4,\tau+1})] + \frac{1}{a}[I_{1,\tau}$$

$$+ \frac{7}{3} I_{3,\tau}]\}[\sin 2g(t) - \sin 2g(t_0)]$$

$$+ \frac{B\beta^4 a^2 \xi^\tau}{3(1-5\theta^2)} \left(\frac{1}{8}[1-11\theta^2 - \frac{40\theta^4}{1-5\theta^2}] - \frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2\right.$$

$$\left. - \frac{8\theta^4}{1-5\theta^2}]\right)\{\tau\xi[eI_{0,\tau+1} + eI_{2,\tau+1}] - \frac{2}{a} I_{1,\tau}\}$$

$$\times [\sin 2g(t) - \sin 2g(t_0)] - \frac{B\beta^6 a^3 \xi^\tau A_{3,0} \sin I}{6(1-5\theta^2)k_2^2}$$

$$\times \{\tau\xi[I_{0,\tau+1} + I_{2,\tau+1} + e(I_{1,\tau+1} + I_{3,\tau+1})]$$

$$+ \frac{3}{a}(I_{0,\tau} + I_{2,\tau})\}[\cos g(t) - \cos g(t_0)],$$

$$G'' = L''\sqrt{1-(e'')^2},$$

$$\Delta e = e'' - e.$$

(6) Quantities $\sin \frac{I}{2} \cos h''$, $\sin \frac{I}{2} \sin h''$, $e'' \cos \ell''$, and $e'' \sin \ell''$ are computed for later use in step (7). Equations are:

$$\sin\frac{I}{2}\cos h'' = \cos h''_s\left[\frac{\delta I}{2}\cos\frac{I}{2} + \sin\frac{I}{2}\right]$$

$$- \sin h''_s \sin\frac{I}{2}\delta h,$$

$$\sin\frac{I}{2}\sin h'' = \sin h''_s\left[\frac{\delta I}{2}\cos\frac{I}{2} + \sin\frac{I}{2}\right]$$

$$+ \cos h''_s \sin\frac{I}{2}\delta h,$$

$$e''\cos\ell'' = (e+\Delta e)\cos\ell''_s - e\delta\ell \sin\ell''_s,$$

$$e''\sin\ell'' = (e+\Delta e)\sin\ell''_s + e\delta\ell\cos\ell''_s.$$

(7) ℓ'', h'', g'' are computed. Equations are:

$$\ell'' = \tan^{-1}[e''\sin\ell''/e''\cos\ell''],$$

$$h'' = \tan^{-1}\left[(\sin\frac{I}{2}\sin h'')/(\sin\frac{I}{2}\cos h'')\right],$$

$$g'' = (\ell''+g''+h'') - \ell'' - h''.$$

(8) Finally, ℓ'', g'', h'', L'', G'', H'' are converted to osculating (unprimed) elements ℓ, g, h, L, G, H. This is done in several steps. In steps (a) and (b), variables on the right side of an equation are assumed to be doubly primed at time t.

(a) $\ell+g+h$ and L are computed. Equations are:

$$\ell+g+h = \ell''+g''+h'' + \frac{k_2}{2a^2\beta^4}\sin(2f+2g)\left(\frac{9}{2} + 3\theta - \frac{15}{2}\theta^2\right),$$

$$L = L''\{1 + \frac{k_2}{2a^2}[3\left(\frac{a}{r}\right)^3(1-\theta^2)\cos(2g+2f)$$

$$+ (3\theta^2-1)\left(\left(\frac{a}{r}\right)^3 - \left(\frac{L}{G}\right)^2\right)]\}.$$

(b) The quantities $e\delta\ell$, δI, $(\sin\frac{I}{2})\delta h$, and Δe are computed for later use in step (c). Equations are:

$$e\delta\ell = \frac{a^2 \sin I \cos g}{4k_2}\left[-\frac{A_{3,0}}{a^3} - \frac{5A_{5,0}}{4a^5}\left(-\frac{24\theta^4}{1-5\theta^2} + 1\right.\right.$$

$$\left.\left. - 9\theta^2\right)\right] - \frac{k_2}{a^2\beta^2}\{\sin f\left(1 + \frac{a}{r} + \frac{a^2\beta^2}{r^2}\right)\left(-\frac{1}{2} + \frac{3}{2}\theta^2\right)$$

$$+ \frac{3}{4}(1-\theta^2)[(1 - \frac{a}{r} - \frac{a^2\beta^2}{r^2})\sin(2g+f) + \sin(2g+3f)(\frac{1}{3}$$

$$+ \frac{a}{r} + \frac{a^2\beta^2}{r^2})]\},$$

$$\delta I = \frac{3k_2\theta \sin I \cos(2g+2f)}{2a^2\beta^4} - \frac{e}{\beta^2}\cos I\ \{\frac{a^2}{4k_2}\sin g$$

$$\times\left[\frac{A_{3,0}}{a^3} + \frac{5A_{5,0}}{4a^5\beta^4}\left(1-9\theta^2 - \frac{24\theta^4}{1-5\theta^2}\right)\right]\},$$

$$\Delta e = \frac{a^2}{4k_2}\sin I \sin g\left[\frac{A_{3,0}}{a^3} + \frac{5A_{5,0}}{4a^5\beta^4}\left(-\frac{24\theta^4}{1-5\theta^2}\right.\right.$$

$$\left.\left. + 1 - 9\theta^2\right)\right] + \frac{k_2}{2a^2\beta^4}\{[9(1-\theta^2)\cos f\cos(2g+2f)$$

$$+ (3\theta^2-1)\left(\frac{e}{1+\beta} + e\beta + 3\cos f + \frac{3}{2}e\right)] - (1-\theta^2)[\cos(2g$$

$$+ 3f) + 3\cos(2g+f)]\},$$

$$\sin\frac{I}{2}\delta h = \frac{3k_2\theta\sin(2f+2g)\sin I}{4a^2\cos\frac{I}{2}}.$$

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(c) Quantities $\sin \frac{I}{2} \cos h$, $\sin \frac{I}{2} \sin h$, $e \cos \ell$, and $e \sin \ell$ are computed for use in steps (d) and (e). The equations used are the same as the equations given in step (6) above, except all variables on the left-hand sides are unprimed, and the variables h''_s and ℓ''_s used on the right-hand sides are replaced by ${}^{s}h''$ and ${}^{s}\ell''$ respectively.

(d) ℓ, g, h are computed. The equations used are the same as the equations used in step (7) above, except all variables are unprimed.

(e) H, e, and G are computed. Equations are:

$$H = H'',$$

$$e = \sqrt{(e \cos \ell)^2 + (e \sin \ell)^2},$$

$$G = L\sqrt{1-e^2}.$$