interval can be obtained. This is possible especially for periodic solutions of (1).

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Small Eccentricities or Inclinations in the Brouwer Theory of the Artificial Satellite

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Brouwer's theory of the artificial satellite is re-examined for small eccentricities and inclinations and improved formulas are obtained. The validity of the new formulas is confirmed by comparison with the results of numerical integration.

1. INTRODUCTION

THE solution of the problem of the near-earth satellite, in terms of general perturbation theory, has been given by Brouwer (1959). The resultant formulas for the perturbations of the elements of the orbit are well known to contain singularities both at zero eccentricity constant and at zero inclination constant. As several investigators including Brouwer have remarked, there are no singularities in the coordinates of the satellite and it should be possible to write well-determined expressions for them, in either the small-eccentricity or the small-inclination case. The difficulty with making this approach directly is that it requires the expansion of the coordinates in Taylor series in the element perturbations, and although the first-order terms in these expansions are regular, the higher terms are singular and hence ignoring them is unjustifiable. This is the case with the method employed by Smith (1961). There is an even more fundamental difficulty with the singularities in that the general perturbation theory is based upon an expansion of the Hamiltonian in a Taylor series in the perturbations of the Delaunay variables, and the neglect of higher terms is equally questionable here. Attempts to verify both Brouwer's formalism and Smith's modification, in the small-eccentricity case, by comparison with the results of numerical integration have been unsuccessful. Errors of the first order appear in both, as the eccentricity constant approaches zero.

Since the coordinates are in reality nonsingular, it must be possible to go a step farther: there can be no infinite singularities in the true osculating elements of the orbit. The reason for the apparent presence of such singularities may be clarified by a simple example. Consider a satellite in a potential field which includes the second zonal harmonic, moving in an equatorial circular orbit. It is easy in this case to solve the Cartesian equations of motion for the coordinates in closed form, and to derive from the coordinates and velocity components the true osculating elements. If after this has been done, the coefficient of the second zonal harmonic k_2 is allowed to become zero, the four osculating elements which describe the unperturbed orbit are found to be

$$\begin{gathered} a, \\ e=0, \\ l=0, \\ g+h=n_0t+\text{const.} \end{gathered}$$

On the other hand, Brouwer's theory tacitly takes the elements of the unperturbed orbit in this case to be

$$\begin{gathered} a, \\ e=0, \\ l=n_0t+\text{const}, \\ g+h=\text{const.} \end{gathered}$$

Either set of elements is perfectly capable of describing the unperturbed orbit, but the former set differs from the perturbed elements by small quantities of the order of k_2, as the perturbation theory assumes to be the case, while the differences calculated on the basis of the latter set are of zero order in k_2. The situation is reminiscent of the quantum-mechanical perturbation theory of degenerate stationary states, where the first step in solving the problem is to find the "right linear combination of unperturbed states."

It proves to be possible to avoid all the difficulties with singularities by adopting the well-known procedure of formulating the perturbation theory in terms of Poincaré variables rather than Delaunay variables and calculating from the results the true osculating elements of the perturbed orbit.

2. FORMULATION OF THE PROBLEM

The Hamiltonian, which includes as perturbing terms the zonal harmonics of the earth's potential, is written following Brouwer's notation:

$$F=F_0+F_{1s}+F_{1p}+F_2, \tag{1}$$

where $F_0=\mu^2/2L^2$, F_{1s} and F_{1p} are respectively the secular and short-period terms of order k_2, and F_2 (taken to be of second order in k_2) is the contribution of the zonal harmonics of order higher than two.

The Poincaré canonical variables, in terms of which the perturbation theory is to be formulated, are defined in terms of the Delaunay variables as follows:

$$\begin{aligned}
x_1&=L,\\
x_2&=[2(L-G)]^{\frac{1}{2}}\cos(g+h),\\
x_3&=[2(G-H)]^{\frac{1}{2}}\cos h,\\
y_1&=l+g+h,\\
y_2&=-[2(L-G)]^{\frac{1}{2}}\sin(g+h),\\
y_3&=-[2(G-H)]^{\frac{1}{2}}\sin h.
\end{aligned} \tag{2}$$

These variables have the well-known advantage that the perturbing potential may be written as a power series in x_2, x_3, y_2, and y_3 throughout the range of variation of the latter quantities. Let us for the sake of clarity write the Hamiltonian as $\tilde{F}$ when it is considered as a function of the Poincaré variables and as F when it is considered as a function of the Delaunay variables. Then we have

$$\tilde{F}[x_i(L,G,H,l,g,h),y_i(L,G,H,l,g,h)]\equiv F(L,G,H,l,g,h).$$

$\tilde{F}$ and F thus represent the same dynamical quantity, but as indicated by the tilde they have different functional forms. The same notation will be used for the component parts of the Hamiltonian, as defined in (1), and for the determining function now to be defined.

The determining function is taken to be

$$\tilde{S}=x_1''y_1+x_2''y_2+x_3''y_3+\tilde{S}_1(x_i'',y_i)+\tilde{S}_1^*(x_i'',y_i). \tag{3}$$

$\tilde{S}_1^*$ is defined as the part of the first-order term which is independent of y_1. The new variables are designated by the double prime for eventual agreement with Brouwer's notation. The relations between the primed and unprimed variables for this determining function are

$$\begin{aligned}
x_1&=x_1''+\tilde{S}_{1y_1}, & y_1''&=y_1+\tilde{S}_{1x_1}+\tilde{S}_1{}^*{}_{x_1},\\
x_2&=x_2''+\tilde{S}_{1y_2}+\tilde{S}_1{}^*{}_{y_2}, & y_2''&=y_2+\tilde{S}_{1x_2}+\tilde{S}_1{}^*{}_{x_2},\\
x_3&=x_3''+\tilde{S}_{1y_3}+\tilde{S}_1{}^*{}_{y_3}, & y_3''&=y_3+\tilde{S}_{1x_3}+\tilde{S}_1{}^*{}_{x_3},
\end{aligned} \tag{4}$$

and the Hamilton-Jacobi equation is

$$\tilde{F}^*(x_i'',y_i'')=\tilde{F}(x_i,y_i), \tag{5}$$

the transformation being determined by the condition that $\tilde{F}^*$ shall be independent of y_1''. If $\tilde{F}^*$ is written as $\tilde{F}_0^*+\tilde{F}_1^*+\tilde{F}_2^*$, in powers of k_2, and Eq. (5) is expanded in a Taylor series about the "mixed point" (x_i'',y_i), and the usual separation by orders of magnitude and by type of periodicity is made, the results are

$$\begin{gathered}
\tilde{F}_0^*=\tilde{F}_0,\\
\tilde{F}_1^*=\tilde{F}_{1s},\\
\tilde{F}_{0x_1}\tilde{S}_{1y_1}=\tilde{F}_{1p},
\end{gathered}$$

$$\begin{aligned}
&\tilde{F}_2^*+\tilde{F}_{1sy_2}\tilde{S}_1{}^*{}_{x_2}-\tilde{F}_{1sx_2}\tilde{S}_1{}^*{}_{y_2}+\tilde{F}_{1sy_3}\tilde{S}_1{}^*{}_{x_3}-\tilde{F}_{1sx_3}\tilde{S}_1{}^*{}_{y_3}\\
&\quad=[\tilde{F}_2+\tfrac{1}{2}\tilde{F}_{0x_1x_1}(\tilde{S}_{1y_1})^2+\tilde{F}_{1px_1}\tilde{S}_{1y_1}+\tilde{F}_{1px_2}\tilde{S}_{1y_2}\\
&\qquad+\tilde{F}_{1px_3}\tilde{S}_{1y_3}-\tilde{F}_{1sy_2}\tilde{S}_{1x_2}+\tilde{F}_{1sx_2}\tilde{S}_{1y_2}\\
&\qquad\qquad-\tilde{F}_{1sy_3}\tilde{S}_{1x_3}+\tilde{F}_{1sx_3}\tilde{S}_{1y_3}]_s,
\end{aligned} \tag{6}$$

the subscript s on the bracket in the last equation denoting that terms periodic in y_1 are to be dropped. All of the quantities appearing in this set of equations are functions of (x_i'',y_i) by the derivation, but since they are identities in the x_i'', y_i we may with equal validity take them all as functions of (x_i'',y_i''). Once this change has been made, it is a straightforward process to transform Eqs. (6) into terms of Delaunay variables, the primed Poincaré variables bearing the same functional relationship to the primed Delaunay variables as holds for the two unprimed sets. The point of making such a transformation rather than directly solving Eqs. (6) in terms of Poincaré variables is that the latter course would unnecessarily duplicate a large amount of the work done by Brouwer.

Before considering the transformation of (6), let us note that since $\tilde{F}$ can be written as a power series in x_2, x_3, y_2, and y_3, it is evident from (6) that so also can $\tilde{F}^*$, $\tilde{S}_1$, and $\tilde{S}_1^*$; this being the case, derivatives such as $\tilde{S}_{1x_2}$ and $\tilde{S}_1{}^*{}_{y_3}$ will also be power series and hence regular in behavior at small eccentricity or small inclination. There are therefore no terms in (6) or in (4) which have the potentiality of becoming singular, nor have any such terms been dropped in truncating the expansions, and the expansions may thus be considered valid to the indicated accuracy in the perturbation parameter.

The transformation of the first three of Eqs. (6) into terms of Delaunay variables is simple: the transformation of the fourth is somewhat tedious, but may be eased by invoking the canonical invariance of the Poisson bracket of any two dynamical quantities. The results are

$$\begin{gathered}
F_0^*=F_0,\\
F_1^*=F_{1s},\\
F_{0L}S_{1l}=F_{1p},
\end{gathered}$$

$$\begin{aligned}
F_2^*-F_{1sG}S_{1g}^*=[&F_2+\tfrac{1}{2}F_{0LL}(S_{1l})^2+F_{1pL}S_{1l}\\
&+F_{1pG}S_{1g}+F_{1sG}S_{1g}]_s.
\end{aligned}$$

(In carrying out the transformation, advantage has been taken of the absence of h from the Hamiltonian and the absence of g as well as l and h from F_{1s}.) These are exactly the equations which Brouwer solves for his F^{**}, S_1, and S_1^*. Thus his solutions for these quantities are applicable without change as transformed

Hamiltonian and first-order determining functions in the present theory, and the only remaining deviation from Brouwer is in the transformation generated by the determining function, Eqs. (4).

3. OSCULATING ELEMENTS

Equations (4) remain valid to the desired accuracy if $\tilde{S}_1(x_i'',y_i'')$ and $\tilde{S}_1^*(x_i'',y_i'')$ are substituted for $\tilde{S}_1(x_i'',y_i)$ and $\tilde{S}_1^*(x_i'',y_i)$, the error made being at most second-order periodic. They may then be readily transformed into terms of Delaunay variables:

$$
\begin{aligned}
L&=L''+S_{1l},\\
l+g+h&=l''+g''+h''-(S_1+S_1^*)_L\\
&\qquad -(S_1+S_1^*)_G-(S_1+S_1^*)_H,\\
[2(L-G)]^{\frac{1}{2}}&\exp[i(g+h)]\\
&=[2(L''-G'')]^{\frac{1}{2}}\exp[i(g''+h'')]\\
&\quad\times\left\{1+\frac{1}{2(L''-G'')}[S_{1l}-(S_1+S_1^*)_g]\right.\\
&\qquad\left.-i[(S_1+S_1^*)_G+(S_1+S_1^*)_H]\right\},\\
[2(G-H)]^{\frac{1}{2}}&\exp(ih)\\
&=[2(G''-H'')]^{\frac{1}{2}}\exp(ih'')\\
&\quad\times\left[1+\frac{1}{2(G''-H'')}(S_1+S_1^*)_g-i(S_1+S_1^*)_H\right].
\end{aligned}
\tag{7}
$$

The partial derivatives of S_1 and S_1^* can be obtained from Brouwer's computational formulas. If we define

$$
\begin{aligned}
\delta l&=-(S_1+S_1^*)_L, & \delta L&=S_{1l},\\
\delta g&=-(S_1+S_1^*)_G, & \delta G&=(S_1+S_1^*)_g,\\
\delta h&=-(S_1+S_1^*)_H,
\end{aligned}
$$

then δl, for example, is the sum of Brouwer's long-period and short-period terms, but is not necessarily the difference between l and l''. Brouwer does not give δL and δG explicitly, but rather δa, δe, and δI, where

$$
\delta L=\frac{L''}{2a''}\delta a,
$$

$$
\delta G=-\frac{L''^2}{G''}e''\delta e+\frac{G''}{2a''}\delta a=G''\tan I''\delta I.
$$

Substitution in Eqs. (7) and some algebraic manipulation leads to formulas more convenient to use:

$$
\begin{aligned}
a&=a''+\delta a,\\
l+g+h&=l''+g''+h''+\delta(l+g+h),\\
e\cos l&=(e''+\delta e)\cos l''-e''\delta l\sin l'',\\
e\sin l&=(e''+\delta e)\sin l''+e''\delta l\cos l'',\\
\sin(\tfrac{1}{2}I)\cos h&=[\sin(\tfrac{1}{2}I'')+\cos(\tfrac{1}{2}I'')\tfrac{1}{2}\delta I]\cos h''\\
&\qquad -\sin(\tfrac{1}{2}I'')\delta h\sin h'',\\
\sin(\tfrac{1}{2}I)\sin h&=[\sin(\tfrac{1}{2}I'')+\cos(\tfrac{1}{2}I'')\tfrac{1}{2}\delta I]\sin h''\\
&\qquad +\sin(\tfrac{1}{2}I'')\delta h\cos h''.
\end{aligned}
\tag{8}
$$

The algorithm for the use of these formulas is then as follows:

(a) Compute a, $l+g+h$, l'', h'', δe, $e''\delta l$, δI, and

$$
\sin(\tfrac{1}{2}I'')\delta h=\sin I''\delta h/(2\cos\tfrac{1}{2}I'')
$$

from Brouwer's formulas for computation, with one deviation. In computing the short-period terms, Brouwer uses l' and g', remarking that the use of l'' and g'' in their place would be equally satisfactory. In the present case, l'' and g'' must be substituted for l' and g', which may be ill-defined. For a numerical program, it is advisable to add together Brouwer's expressions for l, g, and h before computing $l+g+h$, and to use in computing δe the identities

$$
\begin{aligned}
(1/e'')[(a''/r'')^3-\eta^{-3}]&=\eta^{-6}[e''\eta+e''(1+\eta)^{-1}+3\cos f''\\
&\qquad +3e''\cos^2 f''+e''^2\cos^3 f''],\\
(1/e'')[(a''/r'')^3-\eta^{-4}]&=\eta^{-6}[e''+3\cos f''\\
&\qquad +3e''\cos^2 f''+e''^2\cos^3 f''],
\end{aligned}
$$

in order to avoid the occurrence of terms which may become the small difference of two large quantities.

(b) Compute e, I, l, and h from Eqs. (8). These are valid for all eccentricities and inclinations (except inclinations in the neighborhood of π), although if neither the eccentricity nor the inclination is small they have no advantage over Brouwer's formulas, yielding the same results with a little additional work. However, for machine computation the advantage of having one algorithm for all cases outweighs this small disadvantage. The case in which $I\sim\pi$ can probably be handled most efficiently by using the same algorithm with $\pi-I$ substituted for I and $-h$ for h.

(c) Compute the coordinates and velocity components from the osculating elements in the usual way.

4. COMPARISON WITH O. K. SMITH'S RESULTS

Smith defines a quantity e which is neither the osculating eccentricity nor the eccentricity constant, but in a sense an intermediate constructed quantity. This quantity, which will be labeled e_s to avoid confusion, is given in his paper by the formula

$$
e_s=[e''(e''+\delta e)]^{\frac{1}{2}}.
$$

Comparison of the results of Smith's formulas with those of the present theory indicates that this is erroneous. Agreement is obtained only if

$$
e_s=e''+\delta e
$$

is used instead. Since the present theory does not depend upon expansions of doubtful validity, this

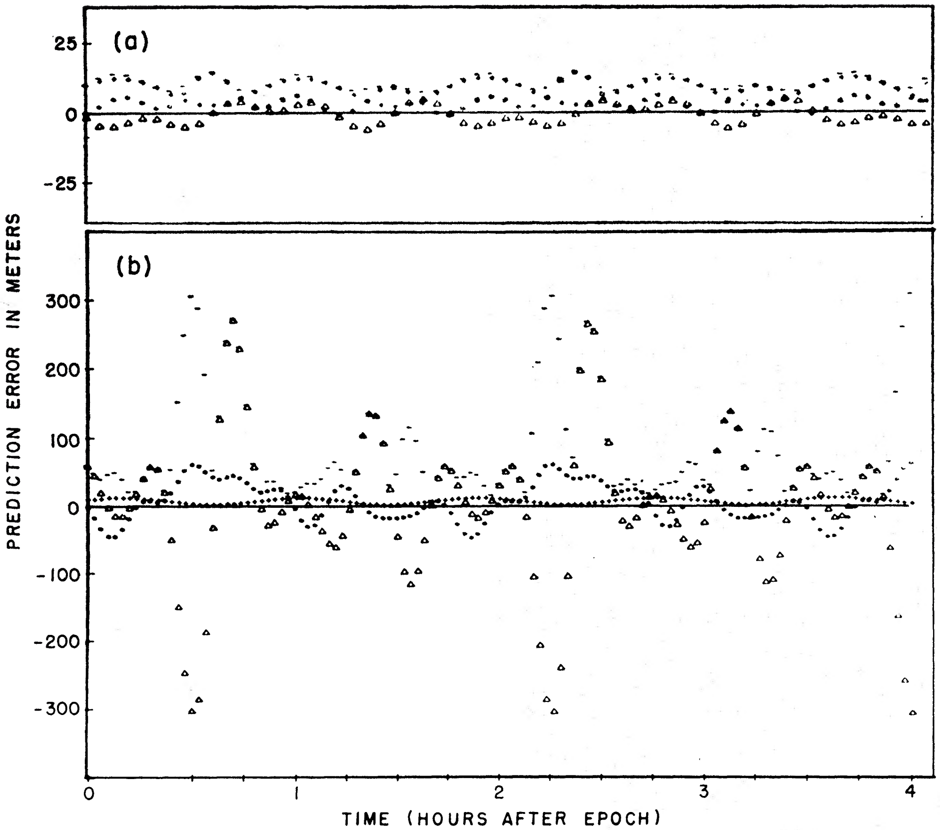

FIG. 1. Coordinate prediction error versus time after epoch. (a) Present theory. (b) Brouwer. Eccentricity constant $e''=0.032$. Coordinate deviations: — total magnitude; * radial component; Δ component along velocity; + component normal to orbit plane.

change will produce accurate results from Smith's formulas. In using these formulas, it should be noted that e_s can assume a negative value. When this occurs, the negative value is to be carried through the calculation; e_s being a constructed quantity need not necessarily be positive. Smith's results for small inclination require no correction.

5. COMPARISON WITH THE RESULTS OF NUMERICAL INTEGRATION

In order to obtain some numerical confirmation of the validity of the foregoing considerations, a comparison was made with the results of a Cowell's integration for the low-eccentricity case. Zonal harmonics through the fifth order were included, and the numerical integration was a tenth-order process with an integration interval of 60 sec and two iterations per time line. All orbits were computed for a 12-h time span.

TABLE I. Maximum coordinate prediction error (meters).

e''	Brouwer	Present theory
0	...	15
0.008	1100	15
0.016	500	15
0.032	300	15

The procedure was first to choose a set of Brouwer constants and calculate the coordinates from Brouwer's formulas. The best fit of a Cowell's orbit to these coordinates was then found by a least-squares determination of the initial coordinates, and the deviations of the Brouwer coordinates from the Cowell coordinates were computed and plotted. The same procedure was repeated, using the same Brouwer constants but the algorithm of the present theory. None of the Brouwer constants was varied except e'', which was assigned several small values; the common values of a'' and I'' were approximately 7365 km and 66°.69, respectively. Typical computer-drawn plots of the coordinate deviations are shown in Fig. 1, and Table I shows maximum coordinate errors taken from these and similar plots.

The errors are periodic in all cases, and since a periodic error of the order of 10 m is to be expected because second-order short-period terms are completely neglected in the general perturbation theory, the agreement between the numerical integration and the present theory is regarded as satisfactory.

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