SPECIAL ASTRODYNAMIC REPORT No. 1

LEVEL II

PROJECT SPACE TRACK

A RESTRICTED FOUR BODY SOLUTION FOR RESONATING SATELLITES WITHOUT DRAG

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A method of averaging is used to study two general problems for the drag free motion of an artificial Earth satellite. First, the four body oblate Earth problem, which describes the satellite motion perturbed by the point mass effects of the Moon and Sun as well as the oblate Earth is analyzed. In this problem the first-order harmonic J_2 and the second-order harmonics J_3 and J_4 are included. Several transformations are introduced and the transformed dynamical system is analytically integrated. Secondly, two specific classes of orbits for the four body oblate Earth problem with resonances due to the sectoral and tesseral potential of the Earth are examined. The transformed dynamical system for this problem is also analytically integrated except for a numerical evaluation of the main resonance effect. Both problems consider satellites with orbital periods up to 24 hours. The resultant model is a mostly analytic, computationally efficient ephemeris generator which includes the most significant perturbations for satellites with orbital periods up to one day.

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1. INTRODUCTION.

Four-body oblate Earth treatments include singly averaged numerical solutions by Kozai [14], Kaula [11], and Cefola [3], singly averaged analytic solutions by Estes [4] and Koskella and Arsenault [13], and doubly averaged numerical solutions by Ash [1] and Lidov [16]. Doubly averaged analytic solutions have not been widely used despite the apparent potential for computational efficiency.

There are very few references which treat the combined problem of resonating satellites in the four body oblate Earth problem. The work of Bowman [2] develops a singly averaged numerical model for 12-h elliptic resonating satellites in the four body oblate Earth problem with drag. A series of publications by Kamel and Tibbits [8], Kamel, Eckman, and Tibbits [9], and Kamel [10] studies a singly averaged, numerically integrated model for 24-h resonating satellites in the four body oblate Earth system by augmenting the solution of Kozai [14]. It is important to note here that a completely analytic solution was obtained by Romanowicz [20,21] for arbitrary resonating satellites in the oblate Earth problem. The Rowanowicz development, however, does not address third body effects.

Large scale space applications and simulations require efficient and compact ephemeris generators which model the major perturbative effects for artificial Earth satellites. This study is a development of such a generator for two related problems for these satellites. The first development applies multiple transformations to the four body oblate Earth problem to obtain an analytic solution. The first-order short periodic variations due to J_2 and the second-order long periodic variations (28 day and 365 day) due to lunar and solar effects are recovered using the transformations introduced by a method of averaging [19]. The second development addresses the problem of resonating satellites in the four body oblate Earth problem. These satellites are assumed to have 12-h or 24-h orbital periods. The method of averaging is again used to derive a multiply transformed dynamical system. The transformed dynamical system is then integrated analytically except for a numerical evaluation of the main resonance effect. The resultant ephemeris generators are computationally efficient. The second model reduces to the first one under non-resonant conditions. For brevity, this discussion is presented in terms of the gravitational functions, and the explicit expressions of the final solution are to be found in the

Appendices. This is a first analysis of the resonant four body oblate Earth problem where multiple transformations are developed.

The result is a model which addresses the same problems as the referenced papers, but does not readily reduce to any particular work.

2. THE FOUR BODY POTENTIAL

The four body oblate Earth problem is a function of the perturbative effects due to the potentials of the Earth, the Moon, and the Sun.

The spherical harmonic potential of the Earth is given by Kaula [12] as

$$V = \sum_{\ell=1}^{\infty} \sum_{m=0}^{\ell} \frac{\mu a_e^{\ell}}{r^{\ell+1}} P_{\ell m}(\sin \xi) \, (C_{\ell m} \cos m\psi$$

$$+ S_{\ell m} \sin m\psi)$$

where $P_{\ell m}$ is the associated Legendre function†. The zonal potential (R_z) can be separated from the potential due to sectoral and tesseral harmonics (R_v) by

$$V = R_z + R_v$$

† The mathematical symbols are defined in the table of notations.

where

$$R_z = \frac{\mu}{r} \left(1 - J_2 \left(\frac{a_e}{r}\right)^2 P_{20} (\sin \xi) - J_3 \left(\frac{a_e}{r}\right)^3 P_{30} (\sin \xi) \right. \tag{1}$$

$$\left. - J_4 \left(\frac{a_e}{r}\right)^4 P_{40} (\sin \xi) + \ldots \right)$$

and R_V is expressed in terms of classical elements by

$$R_V = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\ell} \frac{\mu a_e^{\ell}}{a^{\ell+1}} \sum_{p=0}^{\ell} F_{\ell mp} (i) \sum_{q=-\infty}^{\infty} G_{\ell pq} (e) \, S_{\ell mpq} (\omega, M, \Omega, \theta)$$

where ℓ, m, p, q are integers (2)

$$S_{\ell mpq} (\omega, M, \Omega, \theta) = \begin{bmatrix} C_{\ell m} \\ -S_{\ell m} \end{bmatrix}_{\ell - m \text{ odd}}^{\ell - m \text{ even}} \cos \left[(\ell - 2p)\omega + (\ell - 2p + q)M + m (\Omega - \theta) \right]$$

$$+ \begin{bmatrix} S_{\ell m} \\ C_{\ell m} \end{bmatrix}_{\ell - m \text{ odd}}^{\ell - m \text{ even}} \sin \left[(\ell - 2p)\omega + (\ell - 2p + q)M + m (\Omega - \theta) \right]$$

and $F_{\ell mp}$ (i) and $G_{\ell pq}$ (e) are functions of inclination and eccentricity, respectively (see Kaula [12]).

The gravitational potential of the Moon or Sun is represented by the expanded third-body potential:

$$R_x = Gm_x \left[\frac{1}{r_x} + \frac{r^2}{r_x^3} P_{20} (\cos \phi) + \frac{r^3}{r_x^4} P_{30} (\cos \phi) + \ldots \right] \tag{3}$$

where

$$\begin{aligned} \cos \phi = {} & \cos u \cos U_x \cos (\Omega - H_x) \\ & - \sin u \cos U_x \cos i \sin (\Omega - H_x) \\ & + \cos u \sin U_x \cos l_x \sin (\Omega - H_x) \\ & + \sin u \sin U_x \cos i \cos l_x \cos (\Omega - H_x) \\ & + \sin u \sin U_x \sin i \sin l_x \end{aligned} \tag{4}$$

which can be notationally simplified as (see Appendix H):

$$\cos \phi = A \cos f + B \sin f$$

$$\begin{aligned} A &= X_1 \cos F_x + X_2 \sin F_x \\ B &= X_3 \cos F_x + X_4 \sin F_x \end{aligned} \tag{5}$$

It is assumed here that zonal variations due to J_2 are first-order effects, while all other perturbations are second-order effects. The 24-h limit assumed for satellite orbital period permits a truncation of the lunar and solar potentials:

$$R_x = Gm_x \left[\frac{1}{r_x} + \frac{r^2}{r_x^3} P_{20} (\cos \phi) \right] \tag{6}$$

while the zonal potential R_z is truncated after J_4. The potential for the four-body oblate Earth problem is then represented by:

$$R = R_z + R_v + R_L + R_s$$

The associated perturbed dynamical system is defined by partial differentiation of R and substitution in the Lagrange planetary equations (see Appendix B).

3. THE EQUATIONS OF MOTION.

The complete dynamical system includes the orbital motions of the Moon and Sun as well as the rotational rate of the Earth. In this study, only the mean anomalies of the Moon (γ_L) and Sun (γ_S) and Greenwich sidereal time variable (θ) are assumed to have significant rates and it is assumed that $\dot{I}_x = \dot{H}_x = \dot{G}_x = \dot{E}_x = \dot{A}_x = 0$. Thus the dynamical system takes the form:

$$\begin{aligned}
\dot{a} &= \dot{a}_z + \dot{a}_L + \dot{a}_s + \dot{a}_v \\
\dot{e} &= \dot{e}_z + \dot{e}_L + \dot{e}_s + \dot{e}_v \\
\dot{(i)} &= \dot{(i)}_z + \dot{(i)}_L + \dot{(i)}_s + \dot{(i)}_v \\
\dot{\Omega} &= \dot{\Omega}_z + \dot{\Omega}_L + \dot{\Omega}_s + \dot{\Omega}_v
\end{aligned} \tag{7}$$

$$\dot{\omega} = \dot{\omega}_Z + \dot{\omega}_L + \dot{\omega}_S + \dot{\omega}_V$$

$$\dot{M} = n + \dot{M}_Z + \dot{M}_L + \dot{M}_S + \dot{M}_V \tag{7}$$

$$\dot{\gamma}_L = N_L$$

$$\dot{\gamma}_S = N_S$$

$$\dot{\theta} = \text{constant}$$

where the initial conditons are defined by a_o, e_o, ...,θ_o.

4. THE FOUR BODY OBLATE EARTH PROBLEM.

a. The First Transformation. For non-resonant problems, the dynamical system (7) is a function of two fast variables, M and θ, which satisfy the conditions for application of the method of averaging. Through Leibnitz's Rule, the potential R may be averaged directly:

Define:

$$< g >_x = \frac{1}{2\pi} \int_0^{2\pi} g \, dx$$

Then

$$< R >_{M,\theta} = < R_Z >_M + < R_L >_M + < R_S >_M + < R_V >_{M,\theta}$$

where

$$< R_V >_{M,\theta} = 0$$

and from Liu [17]:

$$< R_Z >_M = \frac{J_2 n^2}{2\beta^3} (1 - \frac{3}{2} \sin^2 i) \tag{8}$$

$$- \frac{3J_4 n^2}{64a^2\beta^7}\left(1 + \frac{3}{2}e^2\right)\left(8 - 40 \sin^2 i + 35 \sin^4 i\right)$$

$$- \frac{3J_3 n^2}{8a\beta^5} e \sin i \left(5 \sin^2 i - 4\right) \sin \omega$$

$$- \frac{15J_4 n^2}{64a^2\beta^7} e^2 \sin^2 i \left(6 - 7 \sin^2 i\right) \cos 2\omega \qquad (8)$$

$$+ \frac{3J_2^2 n^2}{128a^2\beta^7}\left[40 \cos^4 i - 8 \cos^2 i - e^2 \left(5 - 18 \cos^2 i\right.\right.$$

$$\left. + 5 \cos^4 i\right) + 4\beta \left(1 - 3 \cos^2 i\right)^2$$

$$\left. + 2e^2 \left(1 - 16 \cos^2 i + 15 \cos^4 i\right) \cos 2\omega\right],$$

and

$$< R_x >_M = \frac{Gm_x a^2}{4r^3} \left(3A^2 + 3B^2 - 2 + e^2 \left(12A^2 - 3B^2 - 3\right)\right)$$

First-order short periodic variations due to J_2 are given by Liu (Ref 17) and repeated in Appendix F, while the second-order short periodic variations are neglected.

b. The Second and Third Transformation. The averaged dynamical system, which is a function of the singly averaged potential $<R>_{M,\theta}$, is dependent upon the fast angular variable γ_L. The method of averaging is again applied by transforming the potential to remove the mean anomaly of the Moon:

$$<< R >_{M,\theta}>_{\gamma_L} = < R_z >_M + << R_L >_M>_{\gamma_L} + < R_s >_M \tag{9}$$

The doubly averaged dynamical system, due to Eq. (9) is now a function of three angle variables Ω, γ_s, and ω, all of which are nearly the same order. The method of averaging can be used to derive a transformed dynamical system which is independent of Ω, γ_s, and ω. This process would, however, give rise to several singularities in the associated transformations. Each singularity would require a special solution in some small neighborhood of the singularity. Such a piecewise treatment results in multiple sets of solutions and is not compact. Consequently, this development uses the method of averaging to derive a transformed dynamical system which is independent of γ_s alone. The method of averaging is applied to the potential $<<R>_{M,\theta}>_{\gamma_L}$ to remove the periodic dependency upon γ_s:

$$<<< R >_{M,\theta}>_{\gamma_L}>_{\gamma_s} = < R_z >_M + << R_L >_M>_{\gamma_L} + << R_s >_M>_{\gamma_s} \tag{10}$$

where

$$<< R_x >_M >_{\gamma_x} = \frac{Gm_x a^2}{8A_x^3 \, (1-E_x{}^2)^{3/2}} \; (Z_1 + Z_3 - (4 + 6e^2))$$

and

$$Z_1 = 3 \; (X_1^2 + X_3^2 - e^2 \; (4X_1^2 - X_3^2))$$

$$Z_3 = 3 \; (X_2^2 + X_4^2 + e^2 \; (4X_2^2 - X_4^2))$$

The long periodic variations in γ_L and γ_s are define by the method of averaging++ and denoted by:

$$\delta R_x = \frac{Gm_x a^2}{4A_x^3 N_x (1 - E_x^2)^{3/2}} \; (Z_1 F_1 + Z_2 F_2 + Z_3 F_3 + F_4) \qquad (11)$$

where

$$\gamma_x = \gamma_{x_o} + N_x \; (t - t_o)$$

$$F_x = \gamma_x + 2 \; E_x \sin \gamma_x$$

$$F_1 = \frac{1}{2} \sin F_x \cos F_x + \frac{1}{2} \; (F_x - \gamma_x) + \frac{1}{3} \, E_x \sin F_x \cos^2 F_x$$

++ See Ancillary Topics for further discussion.

$$+ \frac{2}{3} E_x \sin F_x$$

$$F_2 = \frac{1}{2} \sin^2 F_x - \frac{1}{3} E_x \cos^3 F_x - \frac{1}{4}$$

$$F_3 = - \frac{1}{2} \sin F_x \cos F_x + \frac{1}{2} (F_x - \gamma_x) + \frac{1}{2} E_x \sin^3 F_x$$

$$F_4 = -(2 + 3e^2) E_x \cos F_x$$

$$Z_2 = 6 (X_1X_2 + X_3X_4 + e^2 (4X_1X_2 - X_3X_4))$$

c. Four-Body Integration. The triply transformed four-body oblate Earth dynamical system for non-resonating satellites has the form (with the < > notation implied):

$$\begin{aligned}
\dot{a} &= 0 \\
\dot{e} &= \dot{e}_z + \dot{e}_L + \dot{e}_s \\
\dot{(i)} &= \dot{(i)}_z + \dot{(i)}_L + \dot{(i)}_s \\
\dot{\Omega} &= \dot{\Omega}_z + \dot{\Omega}_L + \dot{\Omega}_s \\
\dot{\omega} &= \dot{\omega}_z + \dot{\omega}_L + \dot{\omega}_s
\end{aligned} \tag{12}$$

$$\dot{M} = n + \dot{M}_z + \dot{M}_L + \dot{M}_s$$

$$\dot{\gamma}_L = N_L$$

$$\dot{\gamma}_s = N_s$$

$$\dot{\theta} = \text{constant}$$

where the secular rates are presented explicitly in Appendix D for zonal expressions and Appendix E for third body effects.

This system is a function of variables which vary slowly with time. It is assumed that the variable rates in system (12) can be treated as epoch quantities for integration purposes, and that the resultant integration may be valid for an extended period of time. For example:

$$a = a_o$$

$$e = e_o + (\dot{e}_z + \dot{e}_L + \dot{e}_s)_o \, (t-t_o)$$

$$i = i_o + ((\dot{i})_z + (\dot{i})_L + (\dot{i})_s)_o \, (t-t_o)$$

$$\Omega = \Omega_o + (\dot{\Omega}_z + \dot{\Omega}_L + \dot{\Omega}_s)_o \, (t-t_o)$$

$$\omega = \omega_o + (\dot{\omega}_z + \dot{\omega}_L + \dot{\omega}_s)_o \, (t - t_o)$$

$$M = M_o + (n + \dot{M}_z + \dot{M}_L + \dot{M}_s)_o \, (t - t_o)$$

5. RESONANCE PROBLEMS.

A resonance condition exists between M and θ when there exist integers ℓ, m, p, and q such that

$$(\ell - 2p)\dot{\omega} + (\ell - 2p + q)\dot{M} + m(\dot{\Omega} - \dot{\theta}) < \varepsilon \tag{13}$$

where $\varepsilon \leq J_2/4$.

Equation (13) naturally motivates the definition of two new variables λ_1 and λ_2, where:

$$\lambda_1 = M + 2(\Omega - \theta) \quad \text{when } \dot{M} \approx 2\dot{\theta}$$

$$\lambda_2 = \omega + M + \Omega - \theta \text{ when } \dot{M} \approx \dot{\theta}$$

a. The First Transformation. The dynamical system (7) is a function of two fast variables M, and θ, but does not satisfy the conditions for a method of averaging under the resonance conditions $\dot{M} = 2\dot{\theta}$ or $\dot{M} = \dot{\theta}$. A change of variable is suggested:

$$M = \lambda_1 - 2\Omega + 2\theta \text{ or } M = \lambda_2 - \Omega - \omega + \theta \tag{14}$$

which results in a new dynamical system:

$$
\begin{aligned}
\dot{a} &= \dot{a}_z + \dot{a}_L + \dot{a}_s + \dot{a}_v \\
\dot{e} &= \dot{e}_z + \dot{e}_L + \dot{e}_s + \dot{e}_v \\
\dot{(i)} &= \dot{(i)}_z + \dot{(i)}_L + \dot{(i)}_s + \dot{(i)}_v \\
\dot{\Omega} &= \dot{\Omega}_z + \dot{\Omega}_L + \dot{\Omega}_s + \dot{\Omega}_v \\
\dot{\omega} &= \dot{\omega}_z + \dot{\omega}_L + \dot{\omega}_s + \dot{\omega}_v \qquad (15) \\
\dot{\lambda} &= \dot{\lambda}_1 \quad \text{or} \quad \dot{\lambda} = \dot{\lambda}_2 \\
\dot{\gamma}_L &= N_L \\
\dot{\gamma}_s &= N_s \\
\dot{\theta} &= \text{constant}
\end{aligned}
$$

The dynamical system (15) is now a function of one fast variable, θ, and is in a form which satisfies the conditions of the method of averaging. Again the potential is averaged directly;

$$< R(\lambda,\theta) >_\theta \; = \; < R_z(\lambda,\theta) >_\theta \; + \; < R_L(\lambda,\theta) >_\theta \; + \; < R_s(\lambda,\theta) >_\theta$$

$$+ \; < R_v(\lambda,\theta) >_\theta$$

$$= \; < R_z >_M \; + \; < R_L >_M \; + \; < R_s >_M \; + \; < R_v^*(\lambda) >_\theta$$

$$+ \; < R_v^{**}(\lambda,\theta) >_\theta$$

where R_v^* is independent of θ and R_v^{**} is dependent upon θ as a result of the change of variable, Eq. (14). If $Q_{\ell mpq}$ is defined by:

$$Q_{\ell mpq} = (\mu a_e / a^{\ell+1}) \; F_{\ell mp}(i) \; G_{\ell pq}(e) \; \sqrt{(C_{\ell m}^2 + S_{\ell m}^2)}$$

then

$$R_v^* (\lambda_1) = \sum_{\ell,m,p,q} Q_{\ell mpq} \cos \left((\ell-2p)\omega + (\ell-2p+q)\lambda_1 - \lambda_{\ell m} \right)$$

or

$$R_v^* (\lambda_2) = \sum_{\ell,m,p,q} Q_{\ell mpq} \cos \left(-q\omega + (\ell-2p+q)\lambda_2 - \lambda_{\ell m} \right) \tag{16}$$

where

$$\lambda_{\ell m} = \begin{cases} \arctan \; (S_{\ell m}/C_{\ell m}) & (\ell-m) \text{ even} \\ \arctan \; (C_{\ell m}/-S_{\ell m}) & (\ell-m) \text{ odd} \end{cases}$$

In either case, $R_v^*(\lambda)$ is summed over all possible sets of subscripts (ℓ,m,p,q) such that Eq. (13) is satisfied. Therefore

$$R_v^{**}(\lambda,\theta) = R_v(\lambda,\theta) - R_v^*(\lambda) \text{ and}$$

$$< R_v^{**}(\lambda,\theta) >_\theta = 0 \text{ and } < R_v^*(\lambda) >_\theta = R_v^*(\lambda)$$

and

$$< R(\lambda,\theta) >_\theta = < R_z >_M + < R_L >_M + < R_s >_M + R_v^*(\lambda). \qquad (17)$$

is the singly averaged four-body oblate Earth potential under resonance conditions. First-order periodic variations in mean anomaly are recovered as before while other short periodic effects are neglected.

b. <u>The Second and Third Transformations</u>. Again, the singly averaged dynamical system is in a form which permits averaging over γ_L and γ_s successively. The resultant resonating triply averaged potential is given by:

$$<<< R(\lambda,\theta) >_\theta >_{\gamma_L} >_{\gamma_s} = < R_z >_M > + << R_L >_M >_{\gamma_L}$$

$$+ \ll R_S >_M >_{\gamma_S} + R_V^* (\lambda)$$

and the associated dynamical system has the form:

$$
\begin{aligned}
\dot{a} &= \dot{a}_V^* \\
\dot{e} &= \dot{e}_Z + \dot{e}_L + \dot{e}_S \\
(\dot{i}) &= (\dot{i})_Z + (\dot{i})_L + (\dot{i})_S \\
\dot{\Omega} &= \dot{\Omega}_Z + \dot{\Omega}_L + \dot{\Omega}_S \\
\dot{\omega} &= \dot{\omega}_Z + \dot{\omega}_L + \dot{\omega}_S \\
\dot{\lambda} &= \dot{\lambda}_1 \text{ or } \dot{\lambda} = \dot{\lambda}_2 \\
\dot{\gamma}_L &= N_L \\
\gamma_S &= N_S \\
\dot{\theta} &= \text{constant}
\end{aligned}
\qquad (18)
$$

where $\dot{e}_V^*$, $(\dot{i})_V^*$, $\dot{\Omega}_V^*$, $\dot{\omega}_V^*$, and $\dot{M}_V^*$ are neglected. The variable $\dot{a}_V^*$ may be obtained by partial differentiation of $R_V^* (\lambda)$ and substitution into the Lagrange planetary equation for $\dot{a}$.

It is convenient at this time to change variables and replace $\dot{a}$ by

$$\dot{n}_v^* = (-3n/2a)\dot{a}_v^*$$

c. Four-Body Integration. The dynamical system (18) is again a function of variables which vary slowly with time, except n and λ, which may oscillate rapidly. Except for n and λ, the orbital element rates in the system (18) may be treated as epoch quantities for integration purposes. Further, most of the variable expression in n and λ are assumed to be epoch constants, except for n, ω, and λ as follows:

$$\dot{n}_v^* = (\ 3n_o^2 a_o/\mu) \sum_{\ell,m,p,q} (\ell-2p+q)\ (Q_{\ell mpq})_o \sin\,(c\omega + (\ell-2p+q)\,\lambda - \lambda_{\ell m}) \tag{19}$$

$$\dot{\lambda} = \begin{cases} n + (\dot{M}_z + \dot{M}_L + \dot{M}_S + 2(\dot{\Omega}_z + \dot{\Omega}_L + \dot{\Omega}_S) - 2\dot{\theta})_o \\ n + (\dot{M}_z + \dot{M}_L + \dot{M}_S + \dot{\Omega}_z + \dot{\Omega}_L + \dot{\Omega}_S + \dot{\omega}_z + \dot{\omega}_L + \dot{\omega}_S - \dot{\theta})_o \end{cases}$$

where c = -q or $(\ell-2p)$ (see Eq. (16)) and $\omega = \omega_o + \dot{\omega}_o\,(t-t_o)$ for efficiency, a numerical integration of Eq. (19) is chosen, resulting in a mostly analytic integration of the system (18). It should be noted that the argument of perigee must be allowed to vary with time during the integration of Eq. (19), particularly for the highly eccentric 12-h satellites.

6. ANCILLARY TOPICS.

a. Periodic Expressions. The periodic variations due to γ_L and γ_s are recovered in the discussions of second and third transformation. These periodic variations are functions of the integrated transformed variables and are quite complex. To be consistent with the assumptions made in the integration sections, all variables except F_L and F_s are evaluated under epoch conditions. Thus:

$$\delta R_x = \left[\frac{Gm_x a^2}{4A_x^3 N_x \ (1-E_x^2)^{3/2}}\right]_o (Z_{1_o} F_1 + Z_{2_o} F_2 + Z_{3_o} F_3 + F_4) \qquad (20)$$

Both long and short periodic variations are singularity free through a change of variables. An example of such a change of variables is to be found in Appendix G.

b. Small Divisors. The classical element formulation of the secular systems (12) and (18) suffers singularities for zeros in inclination and eccentricity in second order expressions from R_L, R_s, and R_z (J_3 only). These systems can be formulated in terms of equinoctial elements (see Ref (3)). The expressions with classical inclination singularities become third order non-singular expressions in a neighborhood of the singular point. The classical eccentricity singularities occur in the J_3 terms. For high altitude satellites where lunar and solar perturbations are second-order, the perturbations due to J_3 become third-order when eccentricity is small. In either case the expressions with classical singularities become third-order near the singular points and can be neglected in this second-order development. The same result is more economically obtained by simply neglecting the singular classical variable expressions in some small neighborhood of the singularity.

c. The Ephemeris of the Moon and Sun. The classical orbital elements of the Moon and Sun with respect to the equatorial plane at the time of epoch of the satellite elements are obtained using the model presented in the Explanatory Supplement to the Astronomical Ephemeris and American Ephemeris and Nautical Almanac [5]. The model is supplied, without explanation in Appendix I.

d. Other Simplications. Through order of magnitude analysis of the $Q_{\ell mpq}$ coefficients in Eq. (19), a significant reduction in the number of terms in the resonating potential is possible depending upon the desired accuracy. For the existing 24-h satellites, the most significant $Q_{\ell mpq}$ are defined by the set of quadruplets $(\ell,m,p,q) = \{(2,2,0,0),\ (3,1,1,0),\ \text{and}\ (3,3,0,0)\}$. For the existing

12-h satellites, the set of significant quadruplets is much larger due to the ranges of values for inclination and eccentricity. In general, the more significant 12-h quadruplets are those where $\ell \leq 6$ and $m \leq 6$ and Eq. (13) is satisfied. The results obtained for this paper are based upon these simplifications. For NORAD applications the generalized $G_{\ell pq}(e)$ and $F_{\ell mp}(i)$, as presented in Appendix J, proved to be too cumbersome. Explicit expressions have been developed for the most frequently used eccentricity and inclination functions; these explicit functions are also presented in Appendix J.

7. RESULTS.

The graphical results presented here are long-term comparisons versus epoch orbital elements of existing satellites extracted from the NORAD Historical Data System. (Data anomalies due to historical circumstances are not edited). The values of the NORAD elements are represented by X's. The solid lines are predictions with this ephemeris generator, assuming the "X" denoted by "E" as the epoch value. The SAO 1969 geopotential model is used.

Figures 1 and 2 depict the mean motion and east longitude (λ_2) for the 24-h satellite 1971-095B for the period September 1972 through April 1978. Resonating about both "null points", this satellite almost circumnavigates the Earth.

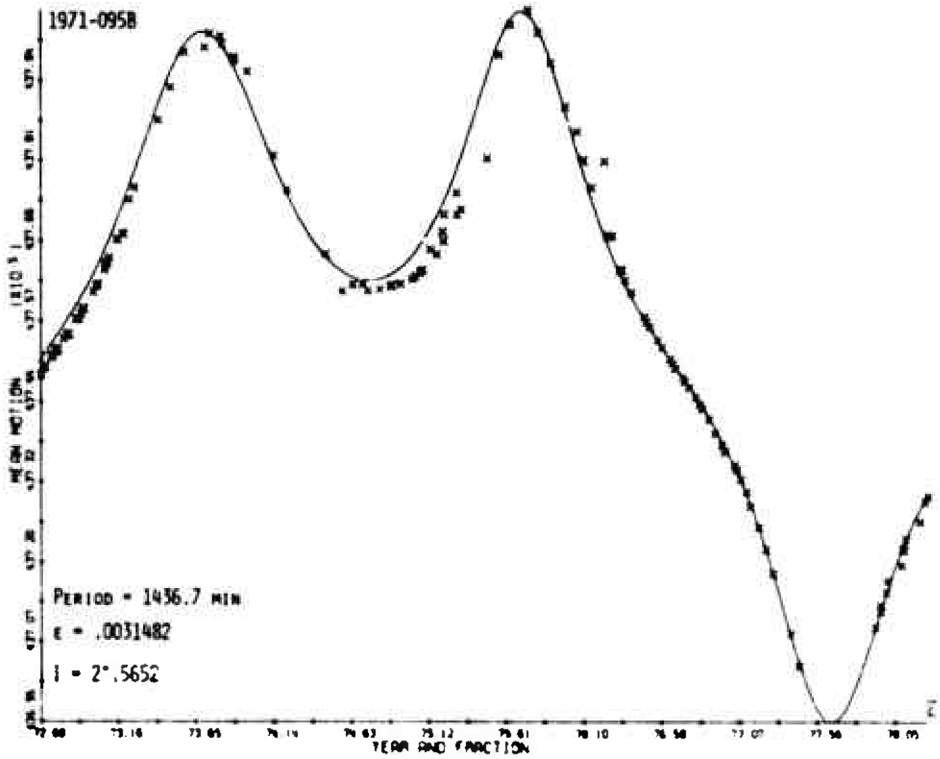

FIG. 1 OBSERVED (x) VS COMPUTED MEAN MOTION (RAD/MIN)

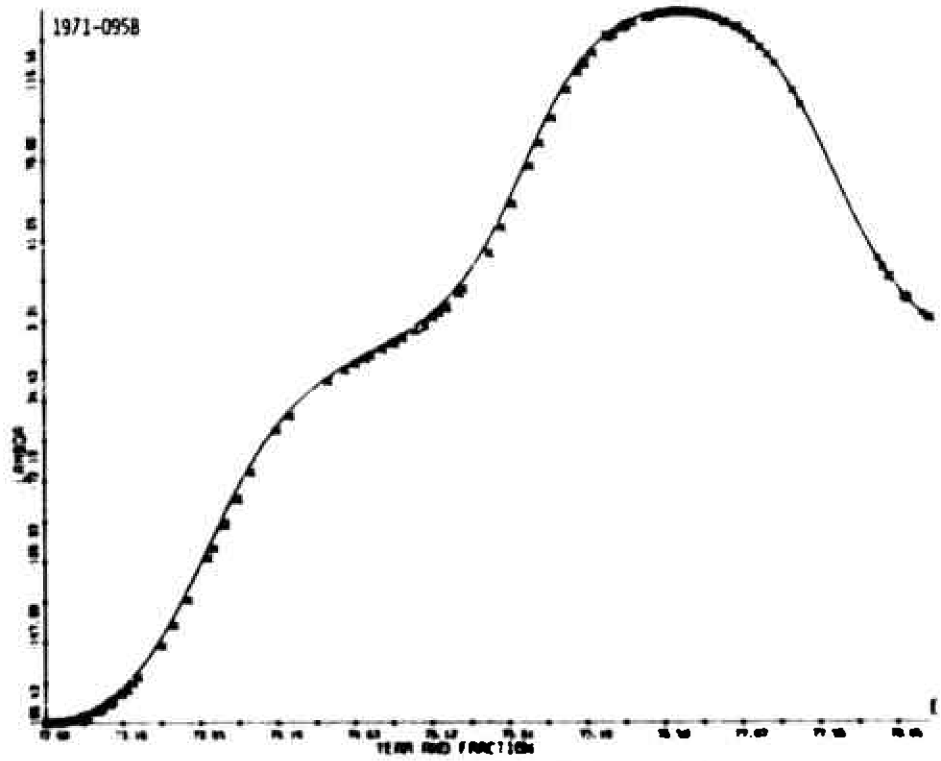

FIG. 2 OBSERVED (x) VS COMPUTED LAMBDA (DEG)

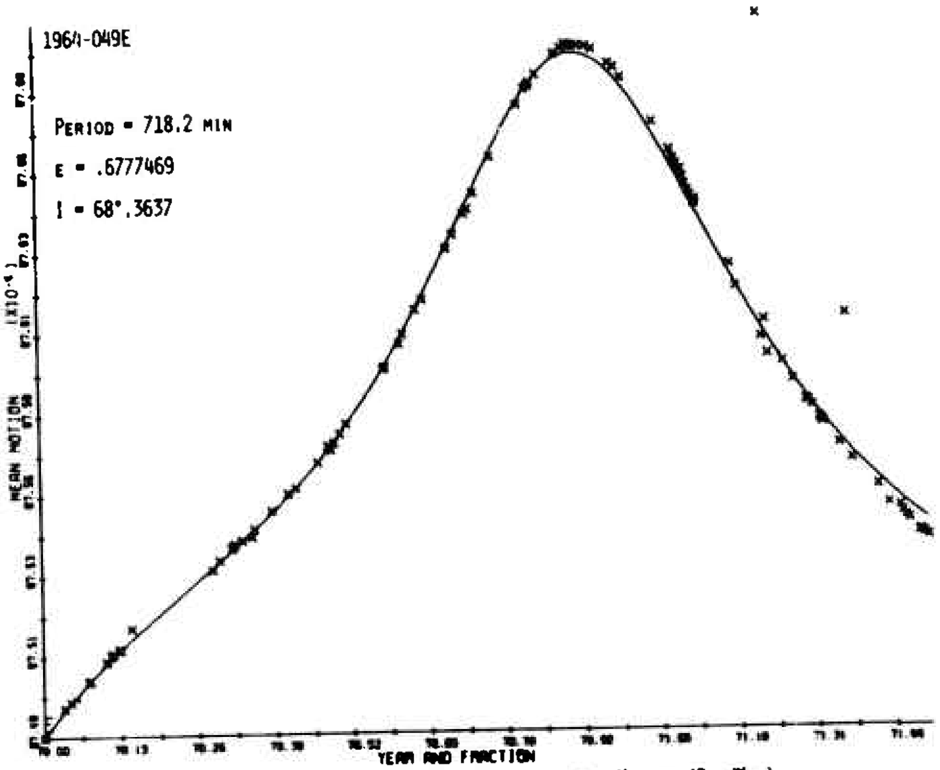

Fig. 3 Observed (x) vs Computed Mean Motion (Rad/Min)

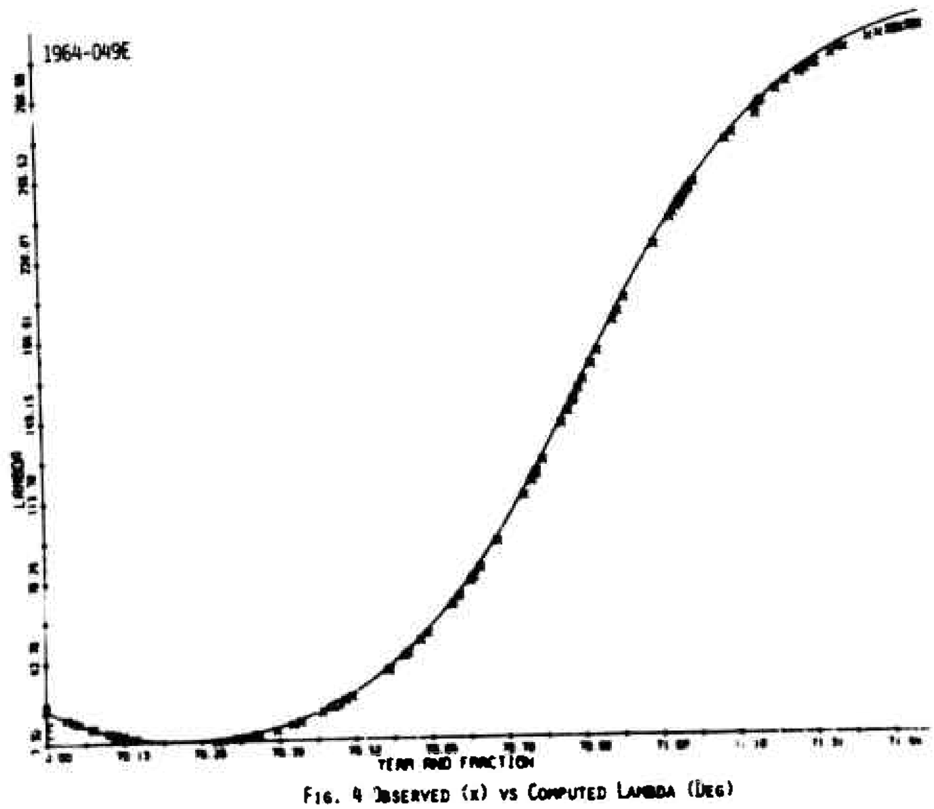

Fig. 4 Observed (x) vs Computed Lambda (Deg)

Figures 3 and 4 present the mean motion and resonance angle (λ_1) for the elliptic 12-h satellite 1964-049E for the period January 1970 through July 1971. This satellite represents an extreme case of resonating 12-h satellites in terms of maximum drift rate in λ_1 (approximately $1\overset{\circ}{.}35$/day).

These two test cases provide a successful examination of this model for resonating satellites in the four-body oblate Earth system.

8. REMARKS.

Throughout the region of interest it is assumed that the zonal effects due to J_2 are first-order variations while all other perturbations are second-order variations. This, of course, may not be valid for cases in which the lunar and solar effects are either larger or smaller than second-order.

For near-Earth satellites, where lunar and solar perturbations are third-order, the reader is referred to one of the many oblate Earth second-order theories.

9. ACKNOWLEDGEMENTS

I wish to express my appreciation to my colleagues in the Aerospace Defense Command Office of Astrodynamics, and in particular to Dr. Felix Hoots and Dr. Joseph Liu for their interest and invaluable assistance in this effort.

10. NOTATION.

All variables are referenced to a non-rotating Earth-centered coordinate system defined by the equatorial plane and the line of poles. In this system, the Sun "appears" to revolve about the Earth and is thus given orbital elements.

Symbol	Definition
A	Intermediate variable
A_x	Semimajor axis of body X
a	Semimajor axis of satellite
a_e	Mean equatorial radius of the Earth
B	Intermediate variable
B_x	$\sqrt{1 - E_x^2}$
$C_{\ell m}$	Cosine coefficient of spherical harmonic potential
E_x	Eccentricity for body X
e	Eccentricity of satellite
F_i	Periodic functions of F_x (see Eq. 11)
F_x	True anomaly of body X
$F_{\ell mp}$	Inclination function
f	True anomaly of satellite
G	Gravitational constant
G_x	Argument of perigee of body X
$G_{\ell pq}$	Eccentricity function

Symbol	Definition
g	Arbitrary function
H_X	Longitude of node of body X
I_X	Inclination of body X
I	Inclination of satellite
J_i	Coefficient of zonal potential term
L	Subscript, referencing the lunar potential
ℓ	Subscript, degree of spherical harmonic
M	Mean anomaly of satellite
m	Subscript, order of spherical harmonic
m_X	Mass of body X
N_X	Mean motion of body X
n	Mean motion of satellite
o	Subscript, evaluated at epoch
$P_{\ell m}$	Legendre associated polynomial
p	Subscript, index of inclination function
$Q_{\ell mpq}$	Amplitude of term in spherical harmonic potential disturbing function
q	Subscript, index for eccentricity function
R	Disturbing function
δR_X	Periodic expression from method of averaging
r	Radial distance to satellite
r_X	Radial distance to body X

Symbol	Definition
s	Subscript, referencing the solar potential
$S_{\ell m}$	Sine coefficient of spherical harmonic potential term
$S_{\ell mpq}$	Combined spherical harmonic potential term
t	Time
U_x	$F_x + G_x$
u	$f + \omega$
v	Subscript, referencing the Earth's harmonic potential
X_i	Intermediate variable
x	Subscript, read L for lunar; s for solar
Z_i	Intermediate variable
β	$\sqrt{1 - e^2}$
θ	Greenwich sidereal time
λ, λ_i	Resonance angle(s)
μ	Gravitational constant of Earth's mass
ξ	Latitude
γ_x	Mean anomaly of body X
ϕ	Phase angle between $\bar{r}$ and $\bar{r}_x$
ψ	Longitude
Ω	Longitude of node of satellite
ω	Argument of perigee of satellite

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APPENDICES

APPENDIX A

A REVIEW OF THE METHOD OF AVERAGING

The following discussion is extracted verbatim from Liu [18] with permission of the author and formally documents the procedures of the Method of Averaging. For a more complete discussion, readers are referred to Morrison [19] or Kyner [15].

Consider a system of first-order ordinary differential equations written in the component form†

$$\dot{x}_i = \varepsilon X_{1i}\,(x_m;\, y_n) + \varepsilon^2 X_{2i}\,(x_m;\, y_n) \qquad \text{(A-1)}$$

$$\dot{y}_j = Z_j(x_m) + \varepsilon Y_{1j}\,(x_m,\, y_n) + \varepsilon^2 Y_{2j}\,(x_m;\, y_n)$$

where $i = 1,2,\ldots,M, j = 1,2,\ldots,N,\ 1 \leq m \leq M$, and $1 \leq n \leq N$ with the initial conditions $x_m(0) = a_m$, $y_n(0) = b_n$. The x_m, are referred to as slow variables because their time variations are proportional to the small parameter ε. The y_n are referred to as fast variables because the dominant parts of their time variations are proportional to t. The functions $X_{1i}\,(x_m;\, y_n)$, $X_{2i}\,(x_m;\, y_n)$, $Y_{1j}\,(x_m,\, y_n)$,

---

† If F_i, $i = 1,2,\ldots L$, are functions of the variables x_m, $m = 1,2,\ldots,M$, we write $F_i(x_m)$. If F_i are functions of variables x_m and y_n, $m = 1,2,\ldots,M, n = 1,2,\ldots,N$, we write $F_i\,(x_m;\, y_n)$

and Y_{2j} $(x_m; y_n)$ are assumed to be continuous functions of x_m and y_n with a period of 2π. In general, the dynamical system, Eqs. (A-1), is non-linear in nature and complex in form and hence the integration of the system is usually analytically intractable.

As the first step in the solution, a transformation is introduced

$$x_i = \bar{x}_i + \varepsilon P_{1i}(\bar{x}_m; \bar{y}_n) + \varepsilon^2 P_{2i}(\bar{x}_m; \bar{y}_n)$$

$$\text{(A-2)}$$

$$y_j = \bar{y}_j + \varepsilon Q_{1j}(\bar{x}_m; \bar{y}_n) + \varepsilon^2 Q_{2j}(\bar{x}_m; \bar{y}_n)$$

so that in a sense the differential equations become simpler to handle in terms of the new variables $\bar{x}_m$ and $\bar{y}_n$. Here $\bar{x}_i$ and $\bar{y}_j$ are regarded as the new unknowns and $P_{1i}(\bar{x}_m; \bar{y}_n)$, $P_{2i}(\bar{x}_m; \bar{y}_n)$, $Q_{1j}(\bar{x}_m; \bar{y}_n)$, $Q_{2j}(\bar{x}_m; \bar{y}_n)$ are periodic functions of each y_n as new functions to be determined in such a way as to effect a simplification in the transformed dynamical system. It is desired that the fast variables $\bar{y}_n$ (to the second-order in ε) be eliminated from the transformed differential equations. Thus, the transformed differential equations are to have the form

$$\dot{\bar{x}}_i = \varepsilon U_{1i}(\bar{x}_m) + \varepsilon^2 U_{2i}(\bar{x}_m) + \varepsilon^3 W_{1i}(\bar{x}_m; \bar{y}_n; \varepsilon)$$

$$\text{(A-3)}$$

$$\dot{\bar{y}}_j = Z_j(\bar{x}_m) + \varepsilon V_{1j}(\bar{x}_m) + \varepsilon^2 V_{2j}(\bar{x}_m) + \varepsilon^3 W_{2j}(\bar{x}_m; \bar{y}_n; \varepsilon)$$

for suitable functions U_{1i}, U_{2i}, V_{1j}, V_{2j}. For a second-order theory, the third-order terms W_{1i} and W_{2j} will be ignored. Thus Eqs. (A-3) become

$$\dot{\bar{x}}_i = \epsilon U_{1i}(\bar{x}_m) + \epsilon^2 U_{2i}(\bar{x}_m)$$

$$\dot{\bar{y}}_j = Z_j(\bar{x}_m) + \epsilon V_{1j}(\bar{x}_m) + \epsilon^2 V_{2j}(\bar{x}_m) \qquad \text{(A-4)}$$

with initial conditions $\bar{x}_m(0) = \bar{a}_m$, $\bar{y}_n(0) = \bar{b}_n$. These initial values are obtained by substituting the initial values a_m, b_n into the transformation (A-2). The explicit expressions of functions U_{1i}, U_{2i}, V_{1j}, V_{2j}, P_{1i}, P_{2i}, Q_{1j}, and Q_{2j} will be given here without proof. For the thorough discussions of the general theory of the method of averaging, see Refs 19 and 15.

To describe the explicit expressions of these functions, the necessary relations and their definitions will be introduced. To begin with, the Fourier series expression for the perturbing function X_{1i} has the form

$$X_{1i}(\bar{x}_m, \bar{y}_n) = X_{1i0}(\bar{x}_m) + X_{1i1}(\bar{x}_m; \bar{y}_n) \qquad \text{(A-5)}$$

where

$$X_{1i0}(\bar{x}_m) = \left(\frac{1}{2\pi}\right)^N \int_0^{2\pi} \cdots \int_0^{2\pi} X_{1i}(\bar{x}_m; \bar{y}_n)\, d\bar{y}_1 \ldots d\bar{y}_N$$

$$X_{1i}(\bar{x}_m;\ \bar{y}_n) = \sum_{\underline{k}} \{ X_{1il\underline{k}c}(\bar{x}_m)\cos[\underline{k},\bar{y}] + X_{1il\underline{k}s}(\bar{x}_m)\sin[\underline{k},\bar{y}]\}$$

$$[\underline{k},\bar{y}] = \sum_{n=1}^{N} k_n\,\bar{y}_n \qquad \text{(A-6)}$$

In Eqs (A-6), the following definitions apply. The notation $\underline{k} = (k_1, k_2, \ldots, k_n, \ldots, k_N)$ is used to denote a vector, each component k_n is an integer; $\underline{k} = \underline{0}$ is not permitted. The notation $\sum_{\underline{k}}$ indicates summation over all possible integer vectors $\underline{k}$.

The notations $X_{1il\underline{k}c}(\bar{x}_m)$ and $X_{1il\underline{k}s}(\bar{x}_m)$ denote the coefficients of the cos $[\underline{k},\bar{y}]$ and sin $[\underline{k},\bar{y}]$ terms, respectively, in the summation. These Fourier-series coefficients are to be determined in the usual way. It can be shown by the method of averaging that

$$U_{1i}(\bar{x}_m) = X_{1i0}(\bar{x}_m), \quad V_{1j}(\bar{x}_m) = Y_{1j0}(\bar{x}_m) \qquad \text{(A-7)}$$

and

$$P_{1i}(\bar{x}_m;\ \bar{y}_n) = -\sum_{\underline{k}} [\underline{k},Z(x_m)]^{-1} \times$$

$$\{X_{1il\underline{k}s}(\bar{x}_m)\cos[\underline{k},\bar{y}] - X_{1il\underline{k}c}(\bar{x}_m)\sin[\underline{k},\bar{y}]\}$$

$$Q_{1j}(\bar{x}_m;\ \bar{y}_n) = -\sum_{\underline{k}} [\underline{k},Z(\bar{x}_m)]^{-1} \times$$

$$\{S_{1j\underline{k}s}(\bar{x}_m)\cos[\underline{k},\bar{y}] - S_{1j\underline{k}c}(\bar{x}_m)\sin[\underline{k},\bar{y}]\}$$

where

$$S_{1j} = \sum_{r=1}^{M} P_{1r}\,(\bar{x}_m;\,\bar{y}_n)\,\frac{\partial Z_j(\bar{x}_m)}{\partial \bar{x}_r} + {}_{1j1}\,(\bar{x}_m;\,\bar{y}_n) \qquad \text{(A-9)}$$

In Eqs. (A-7) and (A-8), the notations $Y_{1j0}(\bar{x}_m)$, $Y_{1j1}\,(\bar{x}_m;\,\bar{y}_n)$, $S_{1j\underline{k}s}(\bar{x}_m)$ and $S_{1j\underline{k}c}(\bar{x}_m)$ have the similar definitions which apply for $X_{1i0}(\bar{x}_m)$, $X_{1i1}(\bar{x}_m;\,\bar{y}_n)$, $X_{1i1\underline{k}s}(\bar{x}_m)$ and $X_{1i1\underline{k}c}(\bar{x}_m)$, respectively. Secondly, two new functions are introduced

$$R_{1i}\,(\bar{x}_m;\,\bar{y}_n) = X_{2i}\,(\bar{x}_m;\,\bar{y}_n) +$$

$$\sum_{r=1}^{M}\left[P_{1r}\,(\bar{x}_m;\,\bar{y}_n)\,\frac{\partial X_{1i}(\bar{x}_m;\,\bar{y}_n)}{\partial \bar{x}_r} - U_{1r}(\bar{x}_m)\,\frac{\partial P_{1i}(\bar{x}_m;\,\bar{y}_n)}{\partial \bar{x}_r}\right] +$$

$$\sum_{s=1}^{N}\left[Q_{1s}\,(\bar{x}_m;\,\bar{y}_n)\,\frac{\partial X_{1i}(\bar{x}_m;\,\bar{y}_n)}{\partial \bar{y}_s} - V_{1s}(\bar{x}_m)\,\frac{\partial P_{1i}(\bar{x}_m;\,\bar{y}_n)}{\partial \bar{y}_s}\right]$$

$$R_{2j}\,(\bar{x}_m;\,\bar{y}_n) = Y_{2j}(\bar{x}_m;\,\bar{y}_n) + \qquad \text{(A-10)}$$

$$\frac{1}{2}\sum_{r=1}^{M}\sum_{w=1}^{M} P_{1r}\,(\bar{x}_m;\,\bar{y}_n)\,P_{1w}\,(\bar{x}_m;\,\bar{y}_n)\,\frac{\partial^2 Z_j(\bar{x}_m)}{\partial \bar{x}_r \partial \bar{x}_w} +$$

$$\sum_{i=1}^{M}\left[P_{1r}\ (\bar{x}_m;\ \bar{y}_n)\ \frac{\partial Y_{1j}(\bar{x}_m;\bar{y}_n)}{\partial \bar{x}_r} - U_{1r}(\bar{x}_m)\ \frac{\partial Q_{1j}(\bar{x}_m;\bar{y}_n)}{\partial \bar{x}_r}\right] +$$

$$\sum_{s=1}^{N}\left[Q_{1s}(\bar{x}_m;\ \bar{y}_n)\ \frac{\partial Y_{1j}(\bar{x}_m;\bar{y}_n)}{\partial \bar{y}_s} - V_{1s}(\bar{x}_m)\ \frac{\partial Q_{1j}(\bar{x}_m;\bar{y}_n)}{\partial \bar{y}_s}\right]$$

The method of averaging shows that

$$U_{2i}(\bar{x}_m) = R_{1i0}(\bar{x}_m),\ V_{2j}(\bar{x}_m) = R_{2j0}(\bar{x}_m) \qquad \text{(A-11)}$$

and

$$P_{2i}\ (\bar{x}_m;\ \bar{y}_n) = -\ \sum_{\underline{k}}\ [\underline{k}, Z(\bar{x}_m)]^{-1}\ x$$

$$\{R_{1i1\underline{k}s}(\bar{x}_m)\cos[\underline{k},\bar{y}] - R_{1i1\underline{k}c}(\bar{x}_m)\sin[\underline{k},\bar{y}]\}$$

$$Q_{2j}\ (\bar{x}_m;\ \bar{y}_n) = -\ \sum_{\underline{k}}\ [\underline{k}, Z(\bar{x}_m)]^{-1}\ x \qquad \text{(A-12)}$$

$$\{S_{2j\underline{k}s}(\bar{x}_m)\cos[\underline{k},\bar{y}] - S_{2j\underline{k}c}(\bar{x}_m)\sin[\underline{k},\bar{y}]\}$$

where

$$S_{2j} = \sum_{r=1}^{M}\ P_{2r}\ (\bar{x}_m;\ \bar{y}_n)\ \frac{\partial Z_j(\bar{x}_m)}{\partial \bar{x}_r} + R_{2j1}\ (\bar{x}_m;\ \bar{y}_n) \qquad \text{(A-13)}$$

Again the notations $R_{1i0}(\bar{x}_m)$, $R_{1i1}\ (\bar{x}_m;\ \bar{y}_n)$, $R_{1i1\underline{k}s}(\bar{x}_m)$, $R_{1i1\underline{k}c}(\bar{x}_m)$; $R_{2j0}(\bar{x}_m)$, $R_{2j1}(\bar{x}_m;\ \bar{y}_n)$, and $S_{2j\underline{k}s}$, $S_{sj\underline{k}c}(\bar{x}_m)$ have the usual meaning in the Fourier series expressions for R_{1i}, R_{2i}, and S_{2j}, respectively. It should also be noted, in arriving at these expressions, that the nonresonant condition, i.e., $[\underline{k}, Z(\bar{x}_m)] \neq 0$, has been assumed.

Following the determination of the functions U's, V's, P's, and Q's, Eqs. (A-4) may either be integrated analytically or integrated numerically with a much longer integration step time than may be used with Eqs. (A-1). The second-order solution for x_i and y_j can then be obtained from Eqs. (A-2).

APPENDIX B

LAGRANGE PLANETARY EQUATIONS

The Lagrange Planetary Equations are presented here for completeness:

$$\dot{a} = \frac{2na^2}{\mu} \frac{\partial R}{\partial M}$$

$$\dot{e} = -\frac{1}{ae}\left(\frac{p^*}{\mu}\right)^{1/2} \frac{\partial R}{\partial \omega} + \frac{np^*}{\mu e} \frac{\partial R}{\partial M}$$

$$(\dot{i}) = -\frac{1}{h \sin i} \frac{\partial R}{\partial \Omega} + \frac{\cot i}{h} \frac{\partial R}{\partial \omega}$$

$$\dot{\Omega} = \frac{1}{h \sin i} \frac{\partial R}{\partial i}$$

$$\dot{\omega} = \frac{h}{\mu a e} \frac{\partial R}{\partial e} - (\cos i)\, \dot{\Omega}$$

$$\dot{M} = n - \frac{np^*}{\mu e} \frac{\partial R}{\partial e} - \frac{2na^2}{\mu}\left(\frac{\partial R}{\partial a}\right)_M$$

where

$$h = na^2\beta$$

$$p^* = a\beta^2$$

$$\beta = \sqrt{1-e^2}$$

APPENDIX C

THE TRUNCATED THIRD BODY POTENTIAL

1. The truncated potential for a third body (x) is given by

$$R_x = Gm_x \frac{r^2}{r_x^3} P_2 (\cos \phi)$$

where

$$\begin{aligned} \cos \phi = {} & \cos u \cos U_x \cos (\Omega - H_x) \\ & - \sin u \cos i \cos U_x \sin (\Omega - H_x) \\ & + \cos u \sin U_x \cos I_x \sin (\Omega - H_x) \\ & + \sin u \cos i \sin U_x \cos I_x \cos (\Omega - H_x) \\ & + \sin u \sin i \sin U_x \sin I_x \end{aligned}$$

2. The averaged potential for a third body is given by:

$$R_x = \frac{Gm_x a^2}{4r_x^3}\left[3A^2 + 3B^2 - 2 + e^2(12A^2 - 3B^2 - 3)\right]$$

where:

$$
\begin{aligned}
A = {} & \cos\omega \cos U_x \cos(\Omega - H_x) \\
& - \sin\omega \cos i \cos U_x \sin(\Omega - H_x) \\
& + \cos\omega \sin U_x \cos I_x \sin(\Omega - H_x) \\
& + \sin\omega \cos i \sin U_x \cos I_x \cos(\Omega - H_x) \\
& + \sin\omega \sin i \sin U_x \sin I_x
\end{aligned}
$$

$$
\begin{aligned}
B = {} & - \sin\omega \cos U_x \cos(\Omega - H_x) \\
& - \cos\omega \cos i \cos U_x \sin(\Omega - H_x) \\
& - \sin\omega \sin U_x \cos I_x \sin(\Omega - H_x) \\
& + \cos\omega \cos i \sin U_x \cos I_x \cos(\Omega - H_x) \\
& + \cos\omega \sin i \sin U_x \sin I_x
\end{aligned}
$$

3. The potential for a third body, averaged first over one period of the satellite, and further averaged over one apparent period of the third body is given by:

$$R_x = \frac{Gm_x a^2}{4A_x^3(1-E_x^2)^{3/2}} \left[\frac{1}{2} Z_1 + \frac{1}{2} Z_3 - (2 + 3e^2) \right]$$

where Z_1, Z_3 are given in Appendix H.

(note that $(1-E_x^2)$ terms are dropped in NORAD applications)

APPENDIX D

ZONAL SECULAR EFFECTS

The singly transformed dynamical system derived by Liu [17,18] is given in terms of classical elements.

$$\dot{a}_z = 0$$

$$\dot{e}_z = -\frac{3}{32} nJ_2^{\,2}\left(\frac{R}{p}\right)^4 \sin^2 i\ (14-15\sin^2 i)\ e\ (1-e^2)\ \sin 2\omega$$

$$-\frac{3}{8} nJ_3\left(\frac{R}{p}\right)^3 \sin i\ (4-5\sin^2 i)\ (1-e^2)\ \cos\ \omega$$

$$-\frac{15}{32} nJ_4\left(\frac{R}{p}\right)^4 \sin^2 i\ (6-7\sin^2 i)\ e\ (1-e^2)\ \sin 2\omega$$

$$(\dot{i})_z = \frac{3}{64} nJ_2^{\,2}\left(\frac{R}{p}\right)^4 \sin 2i\ (14-15\sin^2 i)\ e^2\ \sin 2\omega$$

$$+\frac{3}{8} nJ_3\left(\frac{R}{p}\right)^3 \cos i\ (4-5\sin^2 i)\ e\ \cos\omega$$

$$+\frac{15}{64} nJ_4\left(\frac{R}{p}\right)^4 \sin 2i\ (6-7\sin^2 i)\ e^2\ \sin 2\omega$$

$$\dot{\omega}_z = \frac{3}{4} nJ_2\left(\frac{R}{p}\right)^2 (4-5\sin^2 i) + \frac{3}{16} nJ_2^{\,2}\left(\frac{R}{p}\right)^4 x$$

$$\Big\{ 48-103\sin^2 i + \frac{215}{4}\sin^4 i + \left(7 - \frac{9}{2}\sin^2 i - \frac{45}{8}\sin^4 i\right)e^2$$

$$+ 6\left(1 - \frac{3}{2}\sin^2 i\right)(4-5\sin^2 i)(1-e^2)^{1/2} -$$

$$- \frac{1}{4}[2(14-15\sin^2 i)\sin^2 i - (28-158\sin^2 i$$

$$+ 135\sin^4 i)e^2] \times$$

$$\cos 2\omega\Big\} + \frac{3}{8} nJ_3\left(\frac{R}{p}\right)^3 \left[(4-5\sin^2 i)\frac{\sin^2 i - e^2\cos^2 i}{e\sin i}\right.$$

$$\left. + 2\sin i(13-15\sin^2 i)e\right]\sin\omega - \frac{15}{32} nJ_4\left(\frac{R}{p}\right)^4 \times$$

$$\Big\{16-62\sin^2 i + 49\sin^4 i + \frac{3}{4}(24-84\sin^2 i + 63\sin^4 i)e^2$$

$$+ \left[\sin^2 i(6-7\sin^2 i) - \frac{1}{2}(12-70\sin^2 i\right.$$

$$\left. + 63\sin^4 i)e^2\right]\cos 2\omega\Big\}$$

$$\dot{\Omega}_z = - \frac{3}{2} nJ_2\left(\frac{R}{p}\right)^2 \cos i - \frac{3}{2} nJ_2^{\,2}\left(\frac{R}{p}\right)^4 \cos i\Big\{\frac{9}{4}$$

$$+ \frac{3}{2}(1-e^2)^{1/2}$$

$$- \sin^2 i \left[\frac{5}{2} + \frac{9}{4}(1-e^2)^{1/2}\right] + \frac{e^2}{4}\left(1 + \frac{5}{4}\sin^2 i\right)$$

$$+ \frac{e^2}{8}(7-15\sin^2 i)\cos 2\omega\Big\} - \frac{3}{8} nJ_3\left(\frac{R}{p}\right)^3 x$$

$$(15\sin^2 i - 4)\, e \cot i \sin\omega + \frac{15}{16} nJ_4\left(\frac{R}{p}\right)^4 \cos i\ x$$

$$\left[(4-7\sin^2 i)\left(1 + \frac{3}{2}e^2\right) - (3-7\sin^2 i)\, e^2 \cos 2\omega\right]$$

$$\dot{M}_z = n\left[1 + \frac{3}{2} J_2\left(\frac{R}{p}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)(1-e^2)^{1/2}\right]$$

$$+ \frac{3}{2} nJ_2{}^2\left(\frac{R}{p}\right)^4\Big\{\left(1 - \frac{3}{2}\sin^2 i\right)^2 (1-e^2)$$

$$+ \left[\frac{5}{4}\left(1 - \frac{5}{2}\sin^2 i + \frac{13}{8}\sin^4 i\right) + \frac{5}{8}\left(1 - \sin^2 i\right.\right.$$

$$\left. - \frac{5}{8}\sin^4 i\right) e^2 + \frac{1}{16}\sin^2 i\,(14-15\sin^2 i)\Big(1$$

$$\left.\left. - \frac{5}{2}e^2\right)\cos 2\omega\right](1-e^2)^{1/2}\Big\}$$

$$+ \frac{3}{8} nJ_2{}^2\left(\frac{R}{p}\right)^4 (1-e^2)^{-1/2}\Big\{3\Big[3 - \frac{15}{2}\sin^2 i$$

$$+ \frac{47}{8}\sin^4 i + \left(\frac{3}{2} - 5\sin^2 i + \frac{117}{16}\sin^4 i\right)e^2$$

$$- \frac{1}{8} \left(1 + 5\sin^2 i - \frac{101}{8} \sin^4 i\right) e^4 \Bigg] + \frac{e^2}{8} \sin^2 i \; x$$

$$[70-123\sin^2 i + (56-66\sin^2 i)\; e^2] \cos 2\,\omega$$

$$+ \frac{27}{128} e^4 \sin^4 i \cos 4\,\omega \Bigg\} - \frac{3}{8} nJ_3 \left(\frac{R}{p}\right)^3 \sin i \; (4-5\sin^2 i) \; x$$

$$\frac{1-4e^2}{e} \; (1-e^2)^{1/2} \sin \omega - \frac{45}{128} nJ_4 \left(\frac{R}{p}\right)^4 (8 - 40\sin^2 i$$

$$+ 35\sin^4 i)\; e^2 \sqrt{1-e^2} \quad + \frac{15}{64} nJ_4 \left(\frac{R}{p}\right)^4 \; x$$

$$\sin^2 i \; (6-7\sin^2 i)\; (2-5e^2)\; (1-e^2)^{1/2} \cos 2\,\omega$$

where R is the equatorial radius of the Earth, and $p = a(1-e^2)$

APPENDIX E

THIRD BODY SECULAR EFFECTS

The following equations represent the secular effects due to a third body (subscript x). These expressions can be obtained by partial differentiation of the doubly averaged third body potential as given in Appendix C. As noted in the text, the right hand sides are assumed to be constant functions of the epoch elements of the satellite and the third body. The equational forms of the X_i and Z_j are given in Appendix H.

$$\dot{a}_x = 0$$

$$\dot{e}_x = 15\, C_x n_x \frac{e\beta}{n} (X_1 X_3 + X_2 X_4)$$

$$(\dot{i})_x = \frac{-C_x n_x}{2n\beta} (Z_{11} + Z_{13})$$

$$\dot{M}_x = \frac{-C_x n_x}{n} (Z_1 + Z_3 - 14 - 6e^2)$$

$$(\omega + \Omega \cos i\,)^{\bullet}_x = \frac{C_x n_x \beta}{n} (Z_{31} + Z_{33} - 6)$$

$$(\Omega \sin i)^{\bullet}_x = \frac{C_x n_x}{2n\beta} (Z_{21} + Z_{23})$$

if $i < 3^{\circ}$ set $(\Omega \sin i)^{\bullet}_x = 0$

$$\dot{\Omega}_x = (\Omega \sin i)\dot{}_x / \sin i$$

$$\dot{\omega}_x = (\omega + \Omega \cos i)\dot{}_x - \cos i \, \dot{\Omega}_x$$

APPENDIX F

ZONAL PERIODIC EFFECTS

The first-order short periodic variations due to J_2 are given by Liu [17] and presented here for completeness.

$$\delta a_z = J_2\left(\frac{R^2}{a}\right)\left\{\left(\frac{a}{r}\right)^3\left[\left(1 - \frac{3}{2}\sin^2 i\right) + \frac{3}{2}\sin^2 i \cos 2(\omega+f)\right]\right.$$

$$\left. - \left(1 - \frac{3}{2}\sin^2 i\right)(1-e^2)^{-3/2}\right\}$$

$$\delta e_z = \frac{1}{2} J_2\left(\frac{R}{p}\right)^2\left(1 - \frac{3}{2}\sin^2 i\right)\left\{\frac{1}{e}\left[1 + \frac{3}{2}e^2 - (1-e^2)^{3/2}\right]\right.$$

$$\left. + 3\left(1 + \frac{e^2}{4}\right)\cos f + \frac{3}{2} e \cos 2f + \frac{e^2}{4}\cos 3f\right\}$$

$$+ \frac{3}{8} J_2\left(\frac{R}{p}\right)^2 \sin^2 i\left[\left(1 + \frac{11}{4}e^2\right)\cos(2\omega+f)\right.$$

$$+ \frac{e^2}{4}\cos(2\omega-f) + 5e\cos(2\omega+2f) + \frac{1}{3}\left(7 + \frac{17}{4}e^2\right) x$$

$$\cos(2\omega+3f) + \frac{3}{2} e \cos(2\omega+4f) + \frac{e^2}{4}\cos(2\omega+5f)$$

$$\left. + \frac{3}{2} e \cos 2\omega\right]$$

$$\delta i_z = \frac{3}{8} J_2\left(\frac{R}{p}\right)^2 \sin 2i \; x$$

$$\left[e \cos(2\omega+f) + \cos 2(\omega+f) + \frac{e}{3}\cos(2\omega+3f)\right]$$

$$\delta\Omega_z = -\frac{3}{2} J_2 \left(\frac{R}{p}\right)^2 \cos i \left[f - M + e \sin f - \frac{e}{2}\sin(2\omega+f)\right.$$

$$\left. - \frac{1}{2}\sin 2(\omega+f) - \frac{e}{6}\sin(2\omega+3f)\right]$$

$$\delta\omega_z = \frac{3}{4} J_2 \left(\frac{R}{p}\right)^2 (4 - 5\sin^2 i)(f - M + e \sin f)$$

$$+ \frac{3}{2} J_2 \left(\frac{R}{p}\right)^2 \left(1 - \frac{3}{2}\sin^2 i\right)\left[\frac{1}{e}\left(1 - \frac{1}{4}e^2\right)\sin f\right.$$

$$\left. + \frac{1}{2}\sin 2f + \frac{1}{12} e \sin 3f\right] - \frac{3}{2} J_2 \left(\frac{R}{p}\right)^2 \left\{\frac{1}{e}\left[\frac{1}{4}\sin^2 i\right.\right.$$

$$\left. + \frac{e^2}{2}\left(1 - \frac{15}{8}\sin^2 i\right)\right] \sin(2\omega+f) + \frac{e}{16}\sin^2 i \sin(2\omega-f)$$

$$+ \frac{1}{2}\left(1 - \frac{5}{2}\sin^2 i\right)\sin 2(\omega+f) - \frac{1}{e}\left[\frac{7}{12}\sin^2 i\right.$$

$$\left. - \frac{e^2}{6}\left(1 - \frac{19}{8}\sin^2 i\right)\right]\sin(2\omega+3f)$$

$$\left. - \frac{3}{8}\sin^2 i \sin(2\omega+4f) - \frac{1}{16} e \sin^2 i \sin(2\omega+5f)\right\}$$

$$- \frac{9}{16} J_2 \left(\frac{R}{p}\right)^2 \sin^2 i \sin 2\omega$$

$$\delta M_z = -\frac{3}{2} J_2 \left(\frac{R}{p}\right)^2 \frac{(1-e^2)^{1/2}}{e} \left\{ \left(1 - \frac{3}{2} \sin^2 i\right) x \right.$$

$$\left[\left(1 - \frac{1}{4} e^2\right) \sin f + \frac{e}{2} \sin 2f + \frac{e^2}{12} \sin 3f \right]$$

$$+ \frac{1}{2} \sin^2 i \left[-\frac{1}{2} \left(1 + \frac{5}{4} e^2\right) \sin (2\omega+f) \right.$$

$$- \frac{e^2}{8} \sin (2\omega-f) + \frac{7}{6} \left(1 - \frac{e^2}{28}\right) \sin (2\omega+3f)$$

$$\left. \left. + \frac{3}{4} e \sin (2\omega+4f) + \frac{e^2}{8} \sin (2\omega+5f) \right] \right\} + \frac{9}{16} J_2 \left(\frac{R}{p}\right)^2 (1$$

$$- e^2)^{1/2} \sin^2 i \sin 2\omega$$

where R is the equatorial radius of the Earth and $p = a(1-e^2)$.

APPENDIX G
THIRD BODY PERIODIC EFFECTS

The following equations represent the additive Lunar (second) inverse transformation, as given in Eq. (20) of the text, where X_i, Z_i, F_i and C_L are as given in Appendix H for Lunar-related variables. To obtain the Solar (third) inverse transformation, determine the X_i, Z_i, and C_s for the epoch solar position and F_i at the time of interest and substitute below:

$$\delta e_L = \frac{-30\beta C_L e_o}{n_o} \left(F_1 X_1 X_3 + F_2 (X_2 X_3 + X_1 X_4) + F_3 X_2 X_4\right)$$

$$\delta i_L = \frac{-C_L}{n_o \beta} \left(F_1 Z_{11} + F_2 Z_{12} + F_3 Z_{13}\right)$$

$$\delta M_L = \frac{-2C_L}{n_o} \left(F_1 Z_1 + F_2 Z_2 + F_3 Z_3 + 3E_L \sin F_L \left(-7 - 3 e_o^2\right)\right)$$

$$\delta gh_L = \frac{2C_L \beta}{n_o} \left(F_1 Z_{31} + F_2 Z_{32} + F_3 Z_{33} - 9 E_L \sin F_L\right)$$

$$\delta h_L = \frac{C_L}{n_o \beta} \left(F_1 Z_{21} + F_2 Z_{22} + F_3 Z_{23}\right)$$

The long period third body terms for Ω and ω, when $i_o > .2$ radians, are computed by:

$$\delta\omega_L = \delta gh_L - \cos i_o \; \delta h_L / \sin i_o$$

$$\delta\Omega = \delta h_L / \sin i_o$$

However, as reported in the text, the divisor by $\sin i_o$ for small inclinations necessitates the following change to intermediate variables:

$$\delta\alpha_L = \delta h_L \cos \Omega + \delta i_L \cos i \sin \Omega$$

$$\delta\beta^*_L = \delta h_L \sin \Omega + \delta i_L \cos i \cos \Omega$$

$$\delta L^*_L = \delta \ell_L + \delta gh_L = \delta i_L \Omega \sin i$$

where

$$L^* = M_p + \omega + \Omega \cos (i)$$

$$\alpha = \sin (i) \sin \Omega$$

$$\beta^* = \sin (i) \cos \Omega$$

and where the variables on the right hand sides are integrated, third transformed variables.

In this special case the periodics are applied through the intermediate variables L^*, α, and β^*. The integrated singly transformed variables are denoted by the p subscript:

$$\alpha = \alpha + \delta\alpha_L + \delta\alpha_s$$

$$\beta^* = \beta^* + \delta\beta^*_L + \delta\beta^*_s$$

$$L^* = L^* + \delta L^*_L + \delta L^*_s$$

$$\Omega_p = \tan^{-1} (\alpha/\beta^*)$$

$$i_p = i + \delta i_L + \delta i_s$$

$$M_p = M_p + \delta M_L + \delta M_s$$

$$\omega_p = L^* - M_p - \Omega_p \cos (i_p)$$

$$e_p = e + \delta e_L + \delta e_s$$

$$L_p = M_p + \omega_p + \Omega_p$$

APPENDIX H

INITIAL VARIABLES COMMON TO APPENDICES E AND G

The following variable calculations are intermediate and in themselves have no important physical definition. This collection of terms was chosen to minimize the number of calculations in obtaining third body effects.

$$a_1 = \cos G_{o_x} \cos (\Omega_o - H_{o_x}) + \sin G_{o_x} \cos I_{o_x} \sin (\Omega_o - H_{o_x})$$

$$a_3 = - \sin G_{o_x} \cos (\Omega_o - H_{o_x}) + \cos G_{o_x} \cos I_{o_x} \sin (\Omega_o - H_{o_x})$$

$$a_7 = - \cos G_{o_x} \sin (\Omega_o - H_{o_x}) + \sin G_{o_x} \cos I_{o_x} \cos (\Omega_o - H_{o_x})$$

$$a_8 = \sin G_{o_x} \sin I_{o_x}$$

$$a_9 = \sin G_{o_x} \sin (\Omega_o - H_{o_x}) + \cos G_{o_x} \cos I_{o_x} \cos (\Omega_o - H_{o_x})$$

$$a_{10} = \cos G_{o_x} \sin I_{o_x}$$

$$a_2 = a_7 \cos i_o + a_8 \sin i_o$$

$$a_4 = a_9 \cos i_o + a_{10} \sin i_o$$

$$a_5 = -a_7 \sin i_o + a_8 \cos i_o$$

$$a_6 = -a_9 \sin i_o + a_{10} \cos i_o$$

$$X_1 = a_1 \cos \omega_o + a_2 \sin \omega_o$$

$$X_2 = a_3 \cos \omega_o + a_4 \sin \omega_o$$

$$X_3 = -a_1 \sin \omega_o + a_2 \cos \omega_o$$

$$X_4 = -a_3 \sin \omega_o + a_4 \cos \omega_o$$

$$X_5 = a_5 \sin \omega_o$$

$$X_6 = a_6 \sin \omega_o$$

$$X_7 = a_5 \cos \omega_o$$

$$X_8 = a_6 \cos \omega_o$$

$$Z_{31} = 12X_1^2 - 3X_3^2$$

$$Z_{32} = 24X_1X_2 - 6X_3X_4$$

$$Z_{33} = 12X_2^2 - 3X_4^2$$

$$Z_1 = 3X_1^2 + 3X_3^2 + e_o^2 \, Z_{31}$$

$$Z_2 = 6X_1X_2 + 6X_3X_4 + e_o^2 \, Z_{32}$$

$$Z_3 = 3X_2^2 + 3X_4^2 + e_o^2 \, Z_{33}$$

$$Z_{11} = -6X_1X_7 + 6X_3X_5 + e_o^2 \, (-24X_1X_7 - 6X_3X_5)$$

$$Z_{12} = -6X_1X_8 - 6X_2X_7 + 6X_3X_6 + 6X_4X_5$$

$$+ e_o^2 \, (-24X_2X_7 - 24 X_1X_8 - 6X_3X_6 - 6X_4X_5)$$

$$Z_{13} = -6X_2X_8 + 6X_4X_6 + e_o^2 \, (-24X_2X_8 - 6X_4X_6)$$

$$Z_{21} = 6X_1X_5 + 6X_3X_7 + e_o^2 \, (24X_1X_5 - 6X_3X_7)$$

$$Z_{22} = 6X_2X_5 + 6X_1X_6 + 6X_4X_7 + 6X_3X_8$$

$$+ e_o^2 \, (24X_2X_5 + 24X_1X_6 - 6X_4X_7 - 6X_3X_8)$$

$$Z_{23} = 6X_2X_6 + 6X_4X_8 + e_o^2 \, (24X_2X_6 - 6X_4X_8)$$

Redefine Z_1, Z_2, and Z_3 in terms of Z_i above:

$$Z_i = 2Z_i + (1 - e_o^2) \, Z_{3i} \qquad i = 1,2,3$$

The constants

$$C_L = 4.7968065E\text{-}7 \text{ rad/min}$$

$C_s = 2.98647969E\text{-}6$ rad/min

are derived from

$$C_x = \frac{1}{4} \frac{m_x}{m_x + m_e} N_x$$

where

m_x = mass of perturbing body (Moon or Sun) and

m_e = mass of the Earth

N_x = apparent mean motion of Moon (or Sun) about the Earth

For the Sun we choose to approximate

$$\frac{m_x}{m_x + m_e} \simeq 1,$$

while for the Moon we use $m_x = \frac{1}{81.53} m_e$.

The mean motions of the Sun and Moon are given by (see Appendix I also):

$N_s = 1.19459E\text{-}5$ rad/min

$N_L = 1.583521770E\text{-}4$ rad/min

The variables F_1, F_2, and F_3 are computed as follows where subscript x = L or s for Moon or Sun:

$$\gamma_x = \gamma_{x_o} + \dot{\gamma}_x (t - t_o)$$

$$F_x = \gamma_x + 2E_x \sin \gamma_x$$

$$F_1 = 1/2 \sin F_x \cos F_x$$

$$F_2 = 1/2 \sin^2 F_x - 1/4$$

$$F_3 = -F_1$$

(The F_i presented here are a simplification of the F_i presented in the text (Eq. 11)).

APPENDIX I

LUNAR AND SOLAR EPHEMERIDES

The positions of the Moon and Sun in terms of classical equatorial elements are given as follows (see ref [5]):

$$\Omega_{L_\varepsilon} = 259\overset{\circ}{.}1833275 - 0\overset{\circ}{.}0529539222t^*$$

$$\cos i_{L_o} = \cos \varepsilon \cos i_{L_\varepsilon} - \sin \varepsilon \sin i_{L_\varepsilon} \cos \Omega_{L_\varepsilon}$$

$$\sin i_{L_o} = \sqrt{1 - \cos^2 i_{L_o}}$$

$$m_o^* = 270\overset{\circ}{.}4342 + 13\overset{\circ}{.}1763965268t^*$$

$$\tau_o' = 334\overset{\circ}{.}32955 + 0\overset{\circ}{.}1114040803t^*$$

$$\gamma_{L_o} = m_o^* - \tau_o'$$

$$\sin H_{L_o} \sin i_{L_o} = \sin i_{L_\varepsilon} \sin \Omega_{L_\varepsilon}$$

$$\cos H_{L_o} = \sqrt{1 - \sin^2 H_{L_o}}$$

$$\sin \Delta \sin i_{L_o} = \sin \varepsilon \sin \Omega_{L_\varepsilon}$$

$$\cos \Delta = \cos H_{L_o} \cos \Omega_{L_\varepsilon} + \sin H_{L_o} \sin \Omega_{L_\varepsilon} \cos \varepsilon$$

$$U_{L_o} = m_o^* - \Omega_{L_\varepsilon} + \tan^{-1} \frac{\sin \Delta}{\cos \Delta}$$

$$\gamma_{s_o} = 358\overset{\circ}{.}47584 + 0\overset{\circ}{.}985600267t^*$$

$$N_L = 13\overset{\circ}{.}06499244/\text{day}$$

$$N_s = \overset{\circ}{.}985600267/\text{day}$$

where t* is the time in days since J.D. 241 5020.0 (1900 Jan. 0.5), and

$$i_{L_\varepsilon} = 5\overset{\circ}{.}145396$$

$$\varepsilon = 23\overset{\circ}{.}4441$$

APPENDIX J

INCLINATION AND ECCENTRICITY FUNCTIONS

The generalized eccentricity and inclination formulae, $G_{\ell pq}(e)$ and $F_{\ell mp}(i)$, are given below (see Kaula [12]). These equations are extracted verbatim from Kaula for completeness. For terms in which $\ell - 2p + q = 0$, compute:

$$G_{\ell p(2p-1)}(e) = \frac{1}{(1-e^2)^{\ell-(1/2)}} \sum_{d=0}^{p'-1} \binom{\ell-1}{2d+\ell-2p'} \binom{2d+\ell-2p'}{d} \left(\frac{e}{2}\right)^{2d+\ell-2p'}$$

in which

$p' = p$ for $p \leq \ell/2$,

$p' = \ell - p$ for $p \geq \ell/2$.

For the terms where $\ell - 2p + q \neq 0$, the development of $G_{\ell pq}(e)$ is much more complicated; we merely quote the result of one solution (Tisserand, 1889, p. 256):

$$G_{\ell pq}(e) = (-1)^{|q|} (1 + \beta^2)^{\ell} \beta^{|q|} \sum_{k=0}^{\infty} P_{\ell pqk} Q_{\ell pqk} \beta^{2k}$$

where

$$\beta = \frac{e}{1+\sqrt{1-e^2}}$$

and

$$P_{\ell pqk} = \sum_{r=0}^{h} \binom{2p' - 2\ell}{h - r} \frac{(-1)^r}{r!} \left(\frac{(1-2p' + q')e}{2\beta} \right)^r$$

$h = k + q'$, $q' > 0$; $h = k$, $q' < 0$;

and

$$Q_{\ell pqk} = \sum_{r=0}^{h} \binom{-2p'}{h - r} \frac{1}{r!} \left(\frac{(\ell-2p'+q')e}{2\beta} \right)^r$$

$h = k$, $q' > 0$; $h = k - q'$, $q' < 0$;

$p = p$, $q' = q$ for $p \leq \ell/2$; $p' = \ell-p$, $q' = -q$ for $p > \ell/2$.

While the inclination function $F_{\ell mp}(i)$ is given by

$$F_{\ell mp}(i) = \sum_{t} \frac{(2\ell-2t)!}{t!(\ell-t)!(\ell-m-2t)!2^{2\ell-2t}} \sin^{\ell-m-2t} i$$

$$\times \sum_{s=0}^{m} \binom{m}{s} \cos^s i \sum_{c} \binom{\ell - m - 2t + s}{c}$$

$$\times \binom{m - s}{p - t - c} (-1)^{c-k}$$

Here k is the integer part of $(\ell - m)/2$, t is summed from 0 to the lesser of p or k, and c is summed over all values making the binomial coefficients nonzero.

As discussed in the text, the $G_{\ell pq}(e)$ expressions are much too cumbersome and too computationally slow for a fast general perturbations theory because of the eccentricities with which we are dealing, while the $F_{\ell mp}(i)$ have an exact formulation. The $F_{\ell mp}(i)$ are used as follows:

$$F_{220} = \frac{3}{4} (1 + \cos (i_o))^2$$

$$F_{221} = \frac{3}{2} (\sin (i_o))^2$$

$$F_{321} = \frac{15}{8} \sin (i_o) (1 - 2 \cos (i_o) - 3 \cos^2 (i_o))$$

$$F_{322} = - \frac{15}{8} \sin (i_o) (1 + 2 \cos (i_o) - \cos^2 (i_o))$$

$$F_{441} = \frac{105}{4} \sin^2 (i_o) (1 + \cos (i_o))^2$$

$$F_{442} = \frac{315}{8} \sin^4 (i_o)$$

$$F_{522} = [315/32 (\sin^3 (i_o) - 2 \sin^3 (i_o) \cos (i_o)$$

$$- 5 \sin^3 (i_o) \cos^2 i_o + \sin (i_o) (-2/3$$

$$+ 4/3 \cos(i_o) + 2\cos^2(i_o)))]$$

$$F_{543} = [29.53125 \sin(i_o)\,(-2 - 8\cos(i_o) + 12\cos^2(i_o)$$

$$+ 8\cos^3(i_o) - 10\cos^4(i_o))]$$

and the $G_{\ell pq}(e)$ are approximated by polynomials in e, depending upon the range of value of e. The approximations used by NORAD are as follows:

$$G_{20-1} = .306 - .44\,(e - .64)$$

$$G_{211} = \begin{cases} (3.616 - 13.247e + 16.29e^2) & .5 \le e \le .65 \\ (-72.099 + 331.819e - 508.738e^2 + 266.724e^3) & .65 \le e \le .775 \end{cases}$$

$$G_{310} = \begin{cases} (-19.302 + 117.39e - 228.419e^2 + 156.591e^3) & .5 \le e \le .65 \\ (-346.844 + 1582.851e - 2415.925e^2 + 1246.113e^3) & .65 \le e \le .775 \end{cases}$$

$$G_{322} = \begin{cases} (-18.9068 + 109.7927e - 214.6334e^2 + 146.5816e^3) & .5 \le e \le .65 \\ (-342.585 + 1554.908e - 2366.899e^2 + 1215.972e^3) & .65 \le e \le .775 \end{cases}$$

$$G_{410} = \begin{cases} (-41.122 + 242.694e - 471.094e^2 + 313.953e^3) & .5 \le e \le .65 \\ (-1052.797 + 4758.686e - 7193.992e^2 + 3651.957e^3) & .65 \le e \le .775 \end{cases}$$

$$G_{422} = \begin{cases} (-146.407 + 841.88e - 1629.014e^2 + 1083.435e^3) & .5 \le e \le .65 \\ (-3581.69 + 16178.11e - 24462.77e^2 + 12422.52e^3) & .65 \le e \le .775 \end{cases}$$

$$G_{520} = \begin{cases} (-532.114 + 3017.977e - 5740.032e^2 + 3708.276e^3) & .5 \le e \le .65 \\ (1464.74 - 4664.75e + 3763.64e^2) & .65 \le e \le .715 \\ (-5149.66 + 29936.92e - 54087.36e^2 + 31324.56e^3) & .715 \le e \le .775 \end{cases}$$

$$G_{533} = \begin{cases} (-919.2277 + 4988.61e - 9064.77e^2 + 5542.21e^3) & .5 \le e \le .7 \\ (-37995.78 + 161616.52e - 229838.2e^2 + 109377.94e^3) & .7 \le e \le .78 \end{cases}$$

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A generalized method of averaging is used to study two general problems for drag free motion of an artificial Earth satellite. The first development includes perturbations due to the point masses of the Earth, Moon and Sun as well as zonal variations. The second development extends the first to include tesseral and sectoral harmonics as resonance perturbations for two specific classes of orbits

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